Knowledge Creation and Diffusion with Limited Appropriation*

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Abstract

Innovation is central to economic growth, but so is the diffusion of new knowledge. Such is the finding of recent macro papers that model the interaction between these two forces. Absent in this literature are three key elements that are the focus of this paper. First, we consider the role of frictions in matching innovators and imitators, which mediate the process of knowledge diffusion. Second, we introduce the force of creative destruction, whereby innovators are replaced by imitators. Third, while most of the recent literature has focused on the case where all surplus from knowledge diffusion is captured by the imitators, we consider the full range of surplus shares that the innovators and imitators can appropriate and their impact on growth. In a simple one-period model, we derive a modified Hosios condition for the optimal innovator bargaining weight when firms are homogeneous. But we also find that as the degree of heterogeneity increases, the share of innovators must decrease to maximize growth. Our calibrated dynamic model suggests that the optimal share of surplus that innovators appropriate should lie in the medium range.

Keywords: Innovation, knowledge diffusion, endogenous growth, appropriability

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1 Introduction

Knowledge creation and diffusion are the main forces behind economic growth. Starting with Lucas (2009), a series of recent papers in the macro literature have emphasized the role of diffusion as a contagion process where lagging firms learn the knowledge of more advanced firms in random matching. Moll and Lucas (2014) and Perla and Tonetti (2014) are examples. In these papers the surplus generated is assumed to accrue entirely to the firm on the receiving end. In contrast, many forms of knowledge diffusion allow the transferring firms to appropriate part of the surplus, such as technology licensing deals. Our paper captures this feature by modeling knowledge transfer as a bargaining problem between these two parties. More explicitly, we consider the role of the Nash bargaining weights in this problem and their impact on knowledge creation and diffusion.

Intellectual property rights are a critical part of innovation policy. There is a long literature arguing in favor or against strong patent rights, trading off incentives for innovation and its costs. In our setting, stronger patent rights are associated with a higher bargaining weight for innovators. Our paper considers the impact this has on both incentives for innovation and imitation, and on its overall effect on economic growth. We explore this both theoretically and quantitatively, calibrating a model of innovation and knowledge diffusion.

An increase in the bargaining weight of the innovator, and thus its share of the surplus, has two opposing effects: 1) it directly encourages innovation; and 2) it discourages imitation. In the presence of congestion a la Mortensen and Pissarides (1994), this in turn reduces the contact rate for innovators (increases for imitators), having a negative effect on innovation. At one extreme, when all surplus is appropriated by the firm transferring knowledge, leaving imitators with no surplus, there are no incentives for imitation and knowledge transfer disappears. At the other extreme, when the imitator appropriates all surplus, innovation occurs only for its direct productive benefit to the innovator who disregards the value created by knowledge transfer.

Intellectual property rights play a dual role. On the one hand, they allow innovators to appropriate some of the rents from follow-up firms that use or build upon these innovations. On the other hand, they can discourage competing innovations that follow up and destroy some of the rents obtained by the original innovator, i.e. creative destruction. The relative importance of each of these channels depends on the share of creative destruction as a fraction of total innovation. We find that this plays an important role when considering the impact of intellectual property rights on economic growth.

In a simple one-period model where all firms are ex-ante identical, we show that the maximum level of growth is achieved at an intermediate bargaining weight for innovators that is the one suggested by the well-known Hosios condition (Hosios, 1990) in models of random matching.[1] When firms differ in their initial level of productivity, those above a certain threshold choose to innovate while those below choose to imitate the innovators. We find that as ex-ante heterogeneity increases, the optimal bargaining weight for innovators decreases and becomes zero when heterogeneity is sufficiently high. We also find that the optimal bargaining weight increases with the share of creative destruction, measured by the extent to which adoption by an imitator decreases the original innovator's value.

Heterogeneity matters for two reasons. First, it captures a component of knowledge that is exogenous to a firm's innovation intensity. When this component dominates the component generated by innovation, stronger intellectual property rights, modeled here as a higher innovator bargaining weight, give rents to innovators without substantially changing their innovation or participation decisions, while discouraging imitation at the same time. Second, with ex-ante heterogeneity the marginal firm at the threshold represents a less valuable match for imitators. Thus, the positive external effect on imitators is smaller, while the negative crowding-out effect on innovators is unchanged.

We explore the question quantitatively in a dynamic general equilibrium scenario that is a small variation of the basic setup in Benhabib, Perla and Tonetti (2021) with the addition of matching congestion. There is a fixed set of agents that can either operate as productive firms or imitate others. Each firm's productivity follows a Brownian motion whose drift is governed by costly innovation intensity and whose volatility is exogenous. Imitators contact a randomly selected firm at a rate determined by a matching function. Upon meeting, the two agents split the imitator's entry surplus according to Nash bargaining. With some probability, the original innovator loses the value of its innovation and returns to the pool of imitators. This parameter measures the relative share of creative destruction.

The model is calibrated to match a series of aggregate moments.[2] Given parameter values, we calculate the share of surplus that maximizes growth. As in our simple model, we find that as productivity heterogeneity increases, the optimal innovator share of surplus decreases. In our preliminary calibration, where innovation volatility is chosen

[1] The intuition for this result is as follows. Holding fixed the innovation decision μ, the Hosios condition guarantees that the equilibrium delivers the optimal fraction of innovators in the population, i.e. the one that maximizes total surplus. What is more surprising, is that total surplus when also taking into account the equilibrium choice of μ is also maximized at this point. The intuition behind this result is that in equilibrium innovator profits from innovation are proportional to total surplus, when surplus is maximized so are the total (direct and indirect) incentives for innovation.

[2] These moments include: aggregate growth rate, interest rate, size distribution of firms, volatility of firm size, share of creative destruction (as given by Garcia-Macia, Hsieh and Klenow (2019)) and the ratio of public to private returns to innovation (as given by Bloom, Schankerman and Van Reenen (2013)).

to match moments of the size distribution of firms, the optimal share is in the medium range. Given these parameter values, total growth falls with innovator share mostly because of the severe drop in knowledge diffusion, as the fraction of imitators decreases dramatically with the bargaining weight of innovators. The optimal fraction of innovators is also sensitive to parameters in the matching function. As we vary the elasticity of the matching function from zero to one, the optimal fraction of innovators also goes from zero to one. This shows the importance of matching frictions in the process of knowledge transfer when considering the optimal assignment of intellectual property rights.

1.1 Related Literature

Our paper builds on several strands of literature, including the papers on knowledge diffusion, patent policy and innovation, and matching frictions, as well as recent papers attempting to measure returns to innovation and the extent of creative destruction.

Kortum (1997) considers a setting where firms sample from a fixed distribution of ideas, recognizing that a key to sustained exponential growth is that the stationary part of this distribution has a Pareto upper tail. In Luttmer (2007), this distribution is endogenously generated in the steady state, as entrants learn from the distribution of incumbent's productivity. A similar mechanism is developed in Lucas (2009), where all firms exogenously learn from others and the initial distribution of knowledge has a Pareto tail.[3] Moll and Lucas (2014) and Perla and Tonetti (2014) endogenize the allocation of resources devoted to learning. In these models, resources can be allocated either to production or learning.

Our model builds on König, Lorenz and Zilibotti (2016) and Benhabib et al. (2021), where firms make an optimal choice between innovation and imitation. In a contemporaneous extension of their paper, Benhabib et al. (2021) briefly consider the role of bargaining. Our analysis differs in several respects: First, we focus on the question of intellectual property protection and derive explicitly the bargaining weights implied by patent enforcement. Second, we focus on the relationship between innovator appropriability and growth, and how the growth maximizing policy is affected by several characteristics of the environment, including innovator heterogeneity, matching frictions, and creative destruction. The latter two characteristics are new features we introduce and absent in their model.

We borrow from recent papers that quantify the technology and product market spillover effects of innovation. Mapping our model to their framework, we use Bloom

[3] A class of hybrid models where firm productivity can evolve from the sampling of an exogenous distribution (as in Kortum (1997)) and the distribution of other firms' knowledge (as in Luttmer (2007) and Lucas (2009)) can be found in Alvarez, Buera and Lucas (2008) and Buera and Oberfield (2020).

et al. (2013)'s calculation of the social versus private returns to innovation. Similarly, we use the importance of creative destruction measured by Garcia-Macia et al. (2019).

Our insights on the optimal bargaining weights relate to the literature on matching frictions in random search. Using an alternative protocol relative to Nash bargaining, our expression for the optimal weights resembles Hosios (1990)'s condition, by adjusting for the creative-destruction events. In the presence of heterogeneity, however, this condition fails as pointed out by Shimer and Smith (2001).

2 An Illustrative Model

We first use a simple one-period model to illustrate the key trade-offs in the incentives for innovation and imitation. The primitives of the environment described here carry through to our dynamic model.

2.1 A Search Market for Knowledge Transfers

The economy lasts for one period and consists of a unit mass of firms. Firms are endowed with an initial knowledge denoted by z, which follows a cumulative distribution function $F(z)$. They can engage in either of two activities, which we call innovation or imitation, that result in an end-of-period knowledge z'. Firms obtain a productive value equal to their final knowledge.

A two-stage game takes place, an innovation stage followed by an imitation stage. In the innovation stage, firms choose whether and how much to innovate. They can improve their productivity to $z' = \mu z$ by incurring an investment cost $c(\mu) z$. The innovation cost function is convex and strictly increasing in the innovation intensity μ. The proportion of firms that invest in innovation is denoted by α, and the remaining $1 - \alpha$ of firms wait to imitate.

In the imitation stage, innovators and imitators match randomly in the market for knowledge transfer. The number of matches is given by the matching function $M(\alpha, 1 - \alpha)$, which is increasing in both arguments and has constant returns to scale. Imitators meet a firm randomly selected from the pool of innovators, with a meeting probability $\iota(\alpha) \equiv M/(1 - \alpha)$. Consistently, innovators meet an imitator with probability $\lambda(\alpha) \equiv M/\alpha$. If an imitator meets an innovator of knowledge z', it can copy the innovator's knowledge perfectly; otherwise, its productive value is zero.[4] The technology transfer thus generates

[4] In the dynamic model, we generalize the imitation technology by allowing for imperfect copying.

a surplus of z'. The innovator bargains for a license fee with a bargaining weight β. Thus, the innovator appropriates a share β of the surplus, leaving a share $1-\beta$ to the imitator.

Congestion in Knowledge Diffusion. The matching process in the knowledge transfer market features potential congestion in imitation. If there are more imitators searching for better technologies to copy, it becomes more difficult for them to find such opportunities. This could represent the limited ability of innovators to transfer their knowledge. To illustrate why congestion matters, consider a special case in which the probability of meeting an innovator is constant, and hence independent of the share of innovators, as in the knowledge diffusion models by Perla and Tonetti (2014), Buera and Oberfield (2020), Perla, Tonetti and Waugh (2021), and Benhabib et al. (2021). This situation corresponds to a matching function $M(\alpha, 1-\alpha) = \kappa(1-\alpha)$, where an infinitesimally small mass of innovators can spread their knowledge to the whole economy.

Bargaining Weights and Patent Enforcement. We interpret β as the innovator's bargaining weight in knowledge transfers. The enforcement game below provides one implementation: stronger patent protection in favor of innovators raises β and allows innovators to appropriate a larger share of the rents generated by knowledge diffusion.

We consider the following *enforcement game*. Upon being matched, the innovator makes a take-it-or-leave-it offer to the imitator, asking for a license fee t for the technology transfer. If the imitator rejects the offer and proceeds to copy the technology without obtaining a license, it gets caught with probability β. Once caught, the imitator is excluded from using this technology and receives its outside value of zero. Consequently, the innovator faces a participation constraint for the imitator $z' - t \geq (1-\beta) z'$. This constraint binds when the innovator asks for a license fee $t = \beta z'$.

Equilibrium. Innovators choose innovation intensity μ to maximize their payoff:

$$\max_{\mu} \left[\left(1 + \lambda(\alpha)\beta\right)\mu - c\left(\mu\right) \right] z.$$

The payoff of innovation comes in two parts: a direct payoff through increased output, μ, and an indirect payoff through knowledge transfer, $\beta\lambda(\alpha)\mu$. Their optimal innovation decision equates the marginal private payoff to the marginal cost:

$$1 + \lambda(\alpha)\beta = c'\left(\mu\right). \tag{1}$$

All else being equal, a higher chance of meeting imitators or a higher ability to appropriate rents from knowledge transfers will spur more innovation.

More productive firms find it more appealing to innovate on their own, while less productive firms have more to gain from imitation and, therefore, choose to do so. The marginal firm who is indifferent between innovating or imitating is denoted by a threshold productivity $\underline{z}$, where the resulting share of innovators $\alpha = 1 - F\left(\underline{z}\right)$. This marginal firm is characterized by an indifference condition:

$$\left[\left(1 + \lambda\left(\alpha\right)\beta\right)\mu - c\left(\mu\right)\right]\underline{z} = \iota\left(\alpha\right)\left(1 - \beta\right)\mu\mathbb{E}\left[z \middle| z \geq \underline{z}\right]. \tag{2}$$

Given the bargaining weight β, an equilibrium consists of an allocation $\left\{\alpha, \mu\right\}$ that satisfies equations (1) and (2).[5]

2.2 Optimal Bargaining Weight

There are several sources of externality that may result in suboptimal equilibrium outcomes. One source is incomplete innovator appropriability in knowledge transfers, which can lead to firms exerting too little investment. Another source is congestion externality in the matching process, closely related to those found in labor search models. Both are affected by the innovator's bargaining weight. A higher β improves the incentives to innovate, holding fixed the composition of innovators and imitators in the population. At the same time, it discourages imitation, leading to a higher fraction of innovators and lowering the frequency of knowledge transfers for innovators.

When $\beta = 1$, innovators appropriate all surplus. As a result, the incentives for imitation and knowledge transfer disappear, decreasing the returns to innovation. Conversely, when $\beta = 0$, imitators appropriate all surplus, so innovators gain nothing from knowledge transfers. Innovation is minimized at both extremes. This suggests that, generically, a single policy instrument cannot decentralize the first-best allocation.

To correct for these externalities, the first-best allocation can potentially be achieved with a sufficiently rich set of instruments, for example with an appropriate bargaining weight combined with direct subsidies to innovation. However, as often discussed in the patent literature, direct subsidies might be difficult to implement when innovation is not directly observed (see Kremer, 1998). For these reasons, we ask what can be achieved when the planner has at its disposal only the innovator's bargaining weight. This policy instrument β is chosen to maximize total output net of innovation costs subject to the

[5] The equilibrium is unique and characterized by an interior allocation, i.e., the fraction of innovators $\alpha \in (0, 1)$.

equilibrium incentives for innovation and imitation.

Formally, the planner chooses β, which induces an equilibrium allocation $\{\alpha, \mu\}$, to maximize total output net of innovation costs:

$$\max_{\beta} \quad \underbrace{\alpha \mathbb{E}\left[z | z \geq \underline{z}\right] \left(\mu - c\left(\mu\right)\right)}_{\text{output from innovators net of innovation costs}} \quad + \underbrace{M\left(\alpha, 1-\alpha\right) \mathbb{E}\left[z | z \geq \underline{z}\right] \mu}_{\text{surplus from matching}} \tag{3}$$

where α and μ are the equilibrium allocation satisfying (1) and (2).

In the calculation of (3), the social surplus from knowledge diffusion per match depends not only on the extent of investment μ but also on the selection into innovation–the average innovator knowledge $\mathbb{E}\left[z | z \geq \underline{z}\right]$.

Equation (2) implies that α is a strictly increasing function of β; this defines implicitly β and μ as functions of α.[6] To streamline notation, let $\bar{z} \equiv \mathbb{E}\left[z | z \geq \underline{z}\right]$. Maximizing the planner's objective (3) with respect to α gives:

Proposition 1. *The optimal innovator bargaining weight is characterized by:*

$$\underbrace{\left(\omega - \beta\right) \left(\underline{z} + \frac{\alpha}{1-\alpha} \bar{z}\right)}_{\textit{Congestion}} - \underbrace{\left(1 - \omega\right) \left(\bar{z} - \underline{z}\right)}_{\textit{Displacement}} + \underbrace{\left(1 - \beta\right) \bar{z} \mathcal{E}_{\mu,\alpha}}_{\textit{Innovation Spillover}} = 0, \tag{4}$$

where the matching elasticity $\omega \equiv \frac{\partial \log(M)}{\partial \log(\alpha)}$ *is evaluated with respect to the innovator input of the matching function, and* $\mathcal{E}_{\mu,\alpha} \equiv \frac{d \log \mu(\alpha)}{d \log \alpha}$ *is the elasticity of innovation intensity along the equilibrium locus induced by the choice of* β.

Congestion. The first component in equation (4) is the standard matching wedge behind the Hosios (1990) condition. When the share of innovators changes, the social marginal value of match creation is governed by the matching elasticity ω, whereas the private return to innovators is governed by their surplus share β. This component therefore vanishes when $\beta = \omega$. It is best interpreted as a Hosios or matching-congestion wedge, rather than as the only role played by congestion, because congestion also shapes the quality-displacement force below.

Introducing matching congestion in knowledge diffusion matters and can be crucial for the result. Again it is instructive to consider an economy with no congestion on the imitator's side, and the chance of finding an innovator to copy is independent of the composition of innovators and imitators. This corresponds to maximum congestion on the

[6]This function is defined, increasing and differentiable from α_0 to $\alpha = 1$ where α_0 is the share of innovators when $\beta = 0$.

innovator side, i.e., $\omega = 0$. The matching wedge alone would then call for zero innovator bargaining weight. From equation (2) it also follows that the innovator's meeting rate $\lambda(\alpha)$ must diverge to ∞ as the share of innovators approaches zero, so only an infinitesimally small fraction of innovators are needed to spread the knowledge to the whole economy.

Displacement. The second component in equation (4) is a quality-displacement wedge. The entry of a marginal firm with productivity $\underline{z}$ into the innovator pool lowers the value of knowledge diffusion per match because the marginal innovator is less productive than the average innovator, whose expected knowledge is $\bar{z}$. Reducing the innovator's bargaining weight can deter such low-type innovators and improve the average quality of the knowledge pool. The strength of this wedge depends on the degree of heterogeneity, which we measure by either the gap $\bar{z} - \underline{z}$ or the mean-to-margin ratio $\bar{z}/\underline{z}$.

This displacement effect is mediated by congestion. When matching is uncongested on the innovator side ($\omega = 1$), the entry of a marginal firm creates additional matches without crowding out existing high-quality imitation opportunities, and the displacement wedge vanishes. As innovator-side congestion rises ($\omega < 1$), however, the arrival of a marginal innovator increasingly replaces higher-quality imitation opportunities, amplifying the negative impact of low-type innovators on the average quality of knowledge diffusion.

The matching wedge alone suggests that innovators should be compensated according to their contribution to match creation. The quality-displacement wedge indicates that the surplus division should also reflect their contribution to surplus creation. This modification aligns with the generalized Hosios condition characterized by Mangin and Julien (2021), which they note applies across a wide range of search models.

Innovation Spillover. The third component in equation (4) captures the welfare effect of the induced change in innovation intensity along the equilibrium locus. The bargaining weight β affects innovation in two ways. One is a direct appropriability effect: a higher β lets innovators capture more of the surplus from knowledge transfers, encouraging them to invest. The second is an equilibrium contact-rate effect: a higher β raises the share of innovators α and thereby lowers the contact rate for innovators. The net effect on μ is summarized by $\mathcal{E}_{\mu,\alpha}$ and can be non-monotonic. Innovators internalize only the share β of the diffusion value created by higher innovation intensity, leaving the term $\left(1-\beta\right)\bar{z}$ as the spillover value not captured privately.

The value of β (or equivalently α) that maximizes the induced innovation spillover depends on details of the productivity distribution and innovation technology, except when the mean-to-margin ratio $\bar{z}/\underline{z}$ does not depend on the composition α, as in the

Table 1: Heterogeneity and optimal bargaining weight

	Mean/margin	Optimal β			Innovator	Output
ζ	$\mathbb{E}[z]/\underline{z}$	Congestion	− disp.	+ spill.	share α	diffusion
20	1.05	0.5	0.5	0.5	82%	35%
2	2	0.5	0.28	0.36	47%	57%
1.1	11	0.5	0	0.13	1%	92%

Notes: The matching function is $M(\alpha, 1-\alpha) = \kappa\alpha^{\omega}(1-\alpha)^{1-\omega}$, with $\kappa = 0.5$ and $\omega = 0.5$. The innovation cost function is $c(\mu) = \frac{\chi}{1+\frac{1}{\varepsilon}}\mu^{1+\frac{1}{\varepsilon}}$, with $\chi = 2$ and $\varepsilon = 1$. The one-period example abstracts from creative destruction.

case of a Pareto distribution or when there is no heterogeneity. In such a situation, the innovation spillover is maximized when $\beta = \omega$, as in the Hosios condition.

2.3 Heterogeneity and Displacement: A Pareto Distribution Example

To illustrate the impact of heterogeneity on the optimal bargaining weight, we construct a simple numerical example. The productivity level follows a Pareto distribution: $\forall z \in [z_{\min}, \infty)$, $F(z) = 1 - \left(\frac{z_{\min}}{z}\right)^{\zeta}$, where the tail index $\zeta > 1$. A lower ζ implies a fatter tail in the productivity distribution and a greater extent of heterogeneity. As mentioned earlier, regardless of the number of firms participating in innovation, the average quality of the innovator pool relative to the marginal firm remains constant at $\zeta/(\zeta-1)$.

Table 1 presents the relationship between the optimal innovator bargaining weight and heterogeneity within the economy. The matching function employed in our analysis exhibits a constant elasticity, denoted as $\omega = 0.5$. Our findings indicate that higher levels of heterogeneity are associated with a lower optimal innovator bargaining weight. Specifically, when there is minimal heterogeneity, as observed in the scenario where $\zeta = 20$ and the Pareto tail is thin, the optimal bargaining weight aligns closely with the standard Hosios level. However, as heterogeneity intensifies, the displacement externality becomes more pronounced. Furthermore, when gains derived from the diffusion of existing knowledge are substantial, the optimal bargaining weight experiences a significant downward adjustment. This effect is evident in the specific example where ζ approaches a value close to 1, ultimately driving the optimal bargaining weight down to zero.

Table 2 further illustrates how the displacement effect varies with the degree of congestion. As congestion intensifies, the optimal bargaining weight for innovators increasingly diverges from the value implied by congestion alone. For instance, when $\omega = 0.7$, the optimal weight is approximately 0.65, very close to 0.7; when $\omega = 0.3$, the optimal weight falls to 0.05, substantially below 0.3. This widening gap reflects the strengthening of the

Table 2: Displacement effect and congestion

Congestion ω	Optimal bargaining weight β	Innovator share α	Output diffusion
0.3	0.05	13%	80%
0.5	0.36	47%	57%
0.7	0.65	70%	51%

Notes: The matching function is $M(\alpha, 1-\alpha) = \kappa\alpha^{\omega}(1-\alpha)^{1-\omega}$, with $\kappa = 0.5$. The firms follow a Pareto distribution with a tail index $\zeta = 2$. The innovation cost function is $c(\mu) = \frac{\chi}{1+\frac{1}{\varepsilon}}\mu^{1+\frac{1}{\varepsilon}}$, with $\chi = 2$ and $\varepsilon = 1$. The one-period example abstracts from creative destruction.

displacement effect under more congested matching, as the entry of additional low-type innovators exacerbates quality losses in the pool of imitation opportunities.

These numerical results will also help us to interpret our subsequent results in the dynamic model, where cross-sectional productivity heterogeneity is an endogenous outcome from forces of innovation and diffusion.

3 Dynamic Model

We now build a dynamic model of knowledge creation and diffusion. This dynamic model differs from the static one in that the extent of knowledge heterogeneity is an endogenous outcome of firm innovation and imitation. Furthermore, it embeds the key elements of a workhorse endogenous innovation model and is, therefore, suitable for quantification.

3.1 Environment

Time is infinite and continuous, $t \in [0, \infty)$. The economy consists of a measure-one continuum of potential firms. At each date, a firm is either active or inactive. Active firms produce differentiated intermediate goods and are indexed by productivity z. Let $F(z,t)$ denote the cumulative measure of active firms with productivity no greater than z. The final consumption good is a CES aggregation of the active intermediate goods:

$$Y(t) = \left[\int y(z,t)^{\frac{\eta-1}{\eta}}\, dF(z,t)\right]^{\frac{\eta}{\eta-1}}, \tag{5}$$

where $y(z,t)$ is the output by a firm with productivity z and η is the elasticity of substitution. Therefore, each producer charges the monopoly markup $\frac{\eta}{\eta-1}$.[7]

[7] We assume that if there are competitors the next lowest quality competitor is deterred by zero ex post profit under Bertrand competition.

There is a representative household with preferences for consumption:

$$\int_0^\infty e^{-\rho t} \log \left(C\left(t\right)\right) dt.$$

The representative household supplies labor of amount L and owns all the firms in the economy. The implied interest rate in this economy is thus $r\left(t\right) = \rho + \dot{C}\left(t\right)/C\left(t\right) = \rho + g\left(t\right)$, where $g\left(t\right) \equiv \dot{C}\left(t\right)/C\left(t\right)$.

Own Innovation. Firms can choose to be in either of two states: active or inactive. Active firms produce and invest to improve their existing productivity through innovation. Inactive firms do not produce; they search for better technologies and become active after a successful match. We call active firms *innovators* and inactive firms *imitators*. After a successful match, an imitator becomes an *entrant*.[8]

An innovator firm's productivity follows the Brownian motion:

$$dz = \mu dt + \sigma dB.$$

where the drift μ represents the innovation intensity the firm chooses by allocating entrepreneurial time $c\left(\mu\right)$ to innovation. We assume that $c\left(0\right) = 0$, $c\left(\mu\right)$ is strictly increasing and convex, and $c\left(\mu\right) < 1$ on the relevant domain. Consequently, production uses labor according to

$$y = \left(\left(1 - c\left(\mu\right)\right) e^{z}\right)^{\frac{1}{\eta-1}} \ell.$$

Knowledge Diffusion. Innovators and imitators match randomly in the market for knowledge transfer as described in Section 2.1. Let $\alpha\left(t\right)$ denote the fraction of innovators at time t, where

$$\alpha\left(t\right) = \int_{\underline{z}(t)}^{\infty} dF\left(z,t\right). \tag{6}$$

Given the matching function $M\left(\alpha, 1-\alpha\right)$, the Poisson contact rates for innovators and imitators are $\lambda\left(\alpha\right) \equiv M\left(\alpha, 1-\alpha\right)/\alpha$ and $\iota\left(\alpha\right) \equiv M\left(\alpha, 1-\alpha\right)/(1-\alpha)$, respectively.

When this meeting occurs, the imitator improves the technology z by a quality increment $q \in \left[\underline{q}, \infty\right)$, which is drawn according to the cumulative distribution function $G\left(q\right)$. We assume that G has no atom at zero, so that $\Pr\left(q \geq 0\right) = 1 - G\left(0\right)$. Thus, the imitator's

[8] An alternative interpretation is that there exist a unit mass of researchers that can either be investing in improving their technologies or imitating the existing pool. The dichotomy between productive innovators and non-productive imitators can also be found in the models by Perla and Tonetti (2014) and Benhabib et al. (2021).

productivity jumps to $z+q$. With probability $\delta \in [0,1)$ the knowledge transfer results in "*creative destruction*" of the innovator firm: the imitator learns to produce the same variety; hence, the innovator can be replaced by the imitator and returns to the pool of imitators. Alternatively, with probability $1-\delta$, the knowledge transfer leads to a "*new variety*": the imitator enters with a new variety.

In the event that the imitator learns to produce the same existing variety, the imitator's quality draw needs to be sufficiently good to replace the incumbent, i.e., $q \geq 0$. Thus, the effective creative-destruction probability is $\tilde{\delta} = \delta\left(1-G\left(0\right)\right)$. In contrast, a new-variety knowledge transfer can generate entry whenever the copied and improved technology clears the activity threshold, $z+q \geq \underline{z}\left(t\right)$. To summarize, conditional on entry, the average quality from creative destruction is higher than from new varieties, as in Garcia-Macia et al. (2019). Here, this outcome arises due to competition and selection.

The innovators and imitators bargain for a license fee for the knowledge transfer according to the enforcement game in Section 2. Therefore, the innovators make a take-it-or-leave-it offer to the imitator, asking for a fee equal to a β share of the imitator's entry surplus. The innovator's net payoff also accounts for the displacement loss in creative-destruction matches, as shown in the HJB equation below. Appendix A.2 extends the enforcement game to incorporate the imitation technology used here and shows that both parties will accept this fee.

3.2 Firm's Problem

We first describe the profits firms obtain from their production activities. A firm producing with technology z derives production profit:

$$\pi\left(z,t\right) = \Pi\left(t\right)\left(1-c\left(\mu\right)\right)e^{z},$$

where the aggregate profit level $\Pi\left(t\right) = \frac{1}{\eta} Z\left(t\right)^{\frac{1}{\eta-1}-1} L$ depends on the aggregate productivity in the economy $Z\left(t\right) = \int_{\underline{z}(t)}^{\infty}\left(1-c\left(\mu\left(z,t\right)\right)\right)e^{z}dF\left(z,t\right)$. Further, the final aggregate output is $Y\left(t\right) = Z\left(t\right)^{\frac{1}{\eta-1}} L$. The details of the firm's static profit maximization and aggregation are relegated to Appendix A.3.

Next, we describe firm decisions in their innovation and imitation activities. Let $V\left(z,t\right)$ denote the value of an active innovator firm with productivity z and $W\left(t\right)$ the value of an inactive firm at time t.[9] The expected entry surplus from copying a firm with productivity

[9] The value of an inactive firm does not depend on its previous productivity because inactive firms do not produce and face the same distribution of future matching opportunities. Any former technology is not a payoff-relevant state while the firm is inactive.

z is

$$S^e(z,t) = \delta \int_{q\geq 0} (V(z+q,t) - W(t))dG(q) + (1-\delta) \int_{z+q\geq \underline{z}(t)} (V(z+q,t) - W(t))dG(q). \quad (7)$$

Let

$$\bar{S}^e(t) \equiv \frac{1}{\alpha(t)} \int_{\underline{z}(t)}^{\infty} S^e(z,t)\, dF(z,t)$$

denote the average entry surplus from a randomly contacted innovator.

The innovator value function, $V(z,t)$, and imitator value function, $W(t)$, satisfy the Hamilton-Jacobi-Bellman (HJB) equations:

$$r(t)V(z,t) = \max_{\mu} \Big\{ \Pi(t)\left(1 - c\left(\mu\right)\right) e^z + \mu V_z(z,t) + \frac{1}{2}\sigma^2 V_{zz}(z,t) + V_t(z,t) \quad (8)$$

$$+\lambda(\alpha(t)) \Big[\underbrace{\beta S^e(z,t)}_{\text{Licensing Income}} - \underbrace{\tilde{\delta}\left(V(z,t) - W(t)\right)}_{\text{Creative Destruction Loss}} \Big] \Big\}$$

$$r(t)W(t) = \iota(\alpha(t))\left(1-\beta\right)\bar{S}^e(t) + W_t(t). \quad (9)$$

In equation (8), the flow payoff to the innovator consists of two parts: a direct payoff from production profit and an indirect payoff through knowledge transfers. The expected flow of licensing income is a β share of the entry surplus $S^e(z,t)$, taking into account the arrival rate $\lambda(\alpha(t))$. However, with probability $\tilde{\delta}$, knowledge transfers can lead to firms being replaced by entrants and suffering a loss of $V(z,t) - W(t)$. The production-profit term already nets out the time allocated to innovation through $c\left(\mu\right)$. The drift and diffusion terms, $\mu V_z(z,t)$ and $\frac{1}{2}\sigma^2 V_{zz}(z,t)$, capture the effect of own innovation on the firm's continuation value. The time derivative of the value function accounts for the change of the aggregate state. The firm's innovation decision $\mu(z,t)$ is such that the marginal cost is equal to the marginal benefit from productivity improvement:

$$\Pi(t)\, c'\left(\mu(z,t)\right) e^z = V_z(z,t). \quad (10)$$

In equation (9), the right-hand side computes the flow payoff for imitators: they retain $1-\beta$ share of the average entry surplus from a randomly contacted innovator, adjusting for the contact rate $\iota(\alpha(t))$.

It follows naturally from equation (8) that $V(z,t)$ is increasing in z. As such, there is a threshold $\underline{z}(t)$ such that firms choose to be active if and only if $z \geq \underline{z}(t)$. The activity threshold $\underline{z}(t)$ satisfies the following value-matching and smooth-pasting conditions:

$$V\left(\underline{z}(t), t\right) = W(t) \tag{11}$$

$$V_z\left(\underline{z}(t), t\right) = 0. \tag{12}$$

3.3 Equilibrium Definition

Let $f(z,t)$ denote the density associated with the cumulative measure $F(z,t)$ of active firms. For all $z > \underline{z}(t)$, this density evolves according to the Kolmogorov forward equation (KFE)

$$\begin{aligned} f_t(z,t) =& \lambda(\alpha(t))\left\{\delta \int_0^\infty \left[f\left(z-q,t\right) - f(z,t)\right] dG\left(q\right) + (1-\delta)\int_{\underline{q}}^\infty f\left(z-q,t\right) dG\left(q\right)\right\} \\ & - \frac{\partial}{\partial z}\left(\mu(z,t) f(z,t)\right) + \frac{1}{2}\sigma^2 f_{zz}(z,t), \end{aligned} \tag{13}$$

and, for all $z \leq \underline{z}(t)$, $f(z,t) = 0$.

The distribution of innovator productivity is shaped by forces of innovation and knowledge diffusion. The first term on the right-hand side of equation (13) captures entry and replacement through knowledge diffusion, adjusted by the contact rate $\lambda(\alpha(t))$. In the creative-destruction branch, an entrant with draw $q \geq 0$ replaces an incumbent and reallocates mass from z to $z+q$. In the new-variety branch, an inactive firm enters as an additional active variety after copying an incumbent. The last two terms capture the motion of incumbent innovators along the knowledge scale, with drift given by the innovation intensity $\mu(z,t)$ and variance σ^2.

We now define the equilibrium.

Definition 1 (Equilibrium)**.** An equilibrium consists of value functions $\{S^e(z,t), V(z,t), W(t)\}$, innovation decision rule $\mu(z,t)$, activity threshold $\underline{z}(t)$, fraction of innovators $\alpha(t)$, and active-firm productivity measure $F(z,t)$ with density $f(z,t)$, given the initial productivity distribution $F(z,0)$, such that the following conditions hold:

(i) Static production, pricing, and aggregation satisfy equation (5), the profit equation for $\pi(z,t)$, and the aggregate productivity and output equations for $Z(t)$ and $Y(t)$.

(ii) Household optimality satisfies the Euler equation $r(t) = \rho + \dot{C}(t)/C(t)$, with goods-market clearing $C(t) = Y(t)$.

(iii) Firm values and innovation choices satisfy equations (7), (8), (9), and (10).

(iv) The activity threshold $\underline{z}(t)$ satisfies the value-matching and smooth-pasting conditions (11) and (12).

(v) The fraction of innovators is given by equation (6), and the active-firm productivity density evolves according to the KFE (13), with $f(z,t) = 0$ for all $z \leq \underline{z}(t)$.

4 Balanced Growth Path

This section characterizes the balanced growth path (BGP) of the dynamic economy. Along the BGP, aggregate output and consumption grow at a constant rate, while the productivity distribution is stationary after detrending. We derive the equilibrium growth rate, the upper bound of innovation intensity, and the Pareto tail index of the productivity distribution. We then compare growth accounting in the right tail with growth accounting across the entire distribution, study a perfect-imitation benchmark that isolates the role of knowledge diffusion, and discuss how innovators' appropriability shapes equilibrium growth and heterogeneity.

4.1 Characterization of the BGP

Before characterizing the BGP, we first describe the cross-sectional pattern of innovation among incumbent innovators. The optimality condition (10) implies that innovation intensity is pinned down by the marginal value of improving productivity, $V_z(z,t)$, relative to current productivity e^z. Because an innovator faces the stopping problem summarized by the value-matching and smooth-pasting conditions (11) and (12), firms further above the activity threshold $\underline{z}(t)$ expect to benefit from higher productivity for longer before switching to imitation. As a result, innovation intensity $\mu(z,t)$ increases with productivity z. In the right tail of the productivity distribution, as $z \to \infty$, the likelihood of exiting vanishes, and innovation intensity converges to an upper bound, $\mu(z,t) \to \bar{\mu}(t)$. We formally establish this pattern in Lemma 3 in Appendix A.4.

Along the BGP, aggregate output and consumption grow at a constant rate $g(t) = g$, and the fraction of innovators is constant, $\alpha(t) = \alpha$. Since z is log productivity, the detrended productivity distribution can be transformed into a stationary one,

$$f\left(z + \left(\eta - 1\right)gt, t\right) = f(z). \tag{14}$$

The value functions of innovators and imitators satisfy $V\left(z + \left(\eta - 1\right)gt, t\right) = e^{gt}V(z)$ and

$W(t) = e^{gt}W$. The activity threshold grows additively at rate $(\eta - 1)\, g$,

$$\underline{z}(t) = \underline{z} + (\eta - 1)\, gt.$$

Finally, the innovation upper bound converges to a constant, $\bar{\mu}(t) = \bar{\mu}$.

Assumption 1. *The initial productivity distribution $F(z, 0)$ has bounded support.*

Under Assumption 1, the initial stock of knowledge in the economy is bounded. Sustained growth therefore requires the creation of new knowledge through innovation. The assumption also selects a unique BGP, as in Luttmer (2007).[10]

For later use, define

$$\tilde{\delta} \equiv \delta (1 - G(0)), \qquad s^e \equiv \delta \int_0^\infty e^q dG(q) + (1 - \delta) \int_{\underline{q}}^\infty e^q dG(q).$$

The following proposition characterizes the BGP.

Proposition 2 (BGP)**.** *Under Assumption 1, there exists a unique BGP with the following features.*

(i) Along the BGP, the economy grows at rate g given by

$$g = \frac{1}{\eta - 1} \left\{ \underbrace{\bar{\mu} + \frac{1}{2}\sigma^2 \zeta}_{\text{own innovation}} + \underbrace{\lambda(\alpha) D(\zeta)}_{\text{knowledge diffusion}} \right\}. \tag{15}$$

where

$$D(\zeta) = \frac{1}{\zeta} \left[\underbrace{\delta \int_0^\infty \left(e^{\zeta q} - 1\right) dG(q)}_{\text{creative destruction}} + \underbrace{(1 - \delta) \int_{\underline{q}}^\infty e^{\zeta q} dG(q)}_{\text{new varieties}} \right].$$

[10] If the initial productivity distribution is unbounded, there can potentially exist a continuum of equilibria associated with different growth rates and stationary distributions. For example, Benhabib et al. (2021) distinguishes between a unique growth path and hysteresis, depending on the initial productivity distribution.

(ii) The upper bound of innovation $\bar{\mu}$ is characterized by [11]

$$c'\left(\bar{\mu}\right)=\frac{1-c\left(\bar{\mu}\right)}{\rho+\left(\eta-1\right)g-\bar{\mu}-\frac{1}{2}\sigma^2-\lambda\left(\alpha\right)\left(\beta s^e-\tilde{\delta}\right)}. \tag{16}$$

(iii) The endogenous stationary distribution has an asymptotic Pareto tail with tail index ζ, which satisfies the following equation:

$$\frac{1}{2}\sigma^2+\lambda\left(\alpha\right)D'\left(\zeta\right)=0. \tag{17}$$

The first part of Proposition 2 provides a formula for the growth rate. The terms inside the bracket of Equation (15) capture the shift in the distribution of productivity which is then translated to growth in output dividing by $\eta-1$. The expression decomposes the growth rate into two sources: the speed of innovation by the firms on the knowledge frontier, $\bar{\mu}+\frac{1}{2}\sigma^2\zeta$, and the impact of knowledge diffusion, $\lambda\left(\alpha\right)D\left(\zeta\right)$. In terms of knowledge diffusion, $\lambda\left(\alpha\right)$ is the intensity at which existing technologies are diffused (the contact rate), and the function $D\left(\zeta\right)$, which depends on the Pareto tail index ζ, summarizes the knowledge diffusion force. The knowledge diffusion function $D\left(\zeta\right)$ can be further decomposed into two parts: When an imitator and innovator match, creative destruction occurs with a probability of δ, and a new variety is created with a probability of $1-\delta$. Similar formulas are obtained in continuous-time models in previous papers by Luttmer (2007, 2012) and Benhabib et al. (2021). Note that all components in the formula are endogenous outcomes driven by the incentives for innovation and imitation, including the share of innovators α, the associated matching rate $\lambda\left(\alpha\right)$, and the innovation intensity $\bar{\mu}$. The relative bargaining weights of innovators and imitators affect their incentives and therefore trade off the two sources of growth.

The last part of Proposition 2 characterizes the detrended stationary productivity distribution. This distribution does not have an analytical solution because the innovation intensity $\mu\left(z\right)$ varies with the firm's distance to the activity threshold. However, on the knowledge frontier, as the innovation intensity $\mu\left(z\right)$ converges to its constant upper bound $\bar{\mu}$, we can characterize the distribution in the right tail: it follows an asymptotic Pareto distribution with tail index in equation (17), a useful measure of productivity heterogeneity.

[11] We restrict attention to parameter values for which the denominator in the expression below is positive:

$$\rho+\left(\eta-1\right)g-\bar{\mu}-\frac{1}{2}\sigma^2-\lambda\left(\alpha\right)\left(\beta s^e-\tilde{\delta}\right)>0.$$

This condition ensures that the asymptotic firm value on the BGP is bounded and well-defined.

The variance of innovation outcomes and the diffusion rate together determine the shape of the productivity distribution in the right tail. A higher standard deviation σ leads to a fatter right tail, while a higher diffusion rate $\lambda(\alpha)$ leads to a thinner right tail.

Innovation and knowledge diffusion are two opposing forces in shaping the extent of equilibrium heterogeneity. On the one hand, innovation contributes to firm heterogeneity, stretching out the productivity distribution. This heterogeneity is mainly due to the stochastic nature of innovation outcomes, as captured by the standard deviation of the Brownian motion. Knowledge diffusion, on the other hand, compresses the productivity distribution as the less productive firms are matched with the more productive incumbents. As seen earlier, the speed of renewal is closely related to innovators' contact rate, which explains why increases in $\lambda(\alpha)$ decrease productivity heterogeneity.

4.2 Growth Accounting Across the Distribution

Another growth accounting can be obtained by looking at the entire sample of active firms across the whole distribution, i.e., for all $z \geq \underline{z}$. To compute the aggregate productivity growth, we again look at the three types of innovation and knowledge diffusion events one by one.

We start with own-innovation activities. Following the tail growth accounting that underlies equation (15), we now account for varying levels of innovation intensity across the distribution. For a firm with productivity z choosing innovation intensity $\mu(z)$, its expected growth is $\mu(z) + \frac{1}{2}\sigma^2$. The growth due to own innovation is the weighted average of the individual firm growth rates. Next, creative-destruction events occur at rate $\lambda(\alpha)\delta$ and lead to expected productivity improvement of $\int_0^\infty (e^q - 1)\, dG(q)$. Similarly, new-variety events arrive at rate $\lambda(\alpha)(1-\delta)$. The new varieties have an expected productivity level $\int_{\underline{q}}^\infty e^q dG(q)$ relative to the average incumbent firms. Finally, with the arrival of each new variety, there will be a corresponding exit of an existing variety to balance the composition of incumbents and inactive imitators. Therefore, the exit rate is equal to the new-variety arrival rate. The exiting firms are the marginal firms at the activity threshold $\underline{z}$; therefore, their impact on the growth depends on how large they are relative to the average incumbent, i.e., the *margin-to-mean ratio* $e^{\underline{z}}/\mathbb{E}[e^z]$. Putting these forces together, we obtain the aggregate productivity growth rate:

$$g = \frac{1}{\eta - 1} \left\{ \underbrace{\frac{\mathbb{E}\left[e^{z}\mu\left(z\right)\right]}{\mathbb{E}\left[e^{z}\right]} + \frac{1}{2}\sigma^{2}}_{\text{own innovation}} + \underbrace{\lambda\left(\alpha\right)\delta\int_{0}^{\infty}\left(e^{q}-1\right)dG\left(q\right)}_{\text{creative destruction}} + \underbrace{\lambda\left(\alpha\right)\left(1-\delta\right)\int_{\underline{q}}^{\infty}e^{q}dG\left(q\right)}_{\text{new varieties}} \right. \\ \left. \underbrace{-\lambda\left(\alpha\right)\left(1-\delta\right)\frac{e^{\underline{z}}}{\mathbb{E}\left[e^{z}\right]}}_{\text{exit}} \right\}. \tag{18}$$

The margin-to-mean ratio $e^{\underline{z}}/\mathbb{E}\left[e^{z}\right]$ in the exit component of equation (18) captures the extent of incumbent productivity heterogeneity. This measure encompasses the equilibrium level of knowledge diffusion, the variance of innovation outcomes, and the variation in innovation intensity. This ratio adjusts such that the two growth accounting exercises are aligned, as we show in the following lemma. This insight is more clearly conveyed in an exogenous constant innovation benchmark, $\mu\left(z\right) = \bar{\mu}$, in Online Appendix A.8, a case in which closed forms can be obtained for the whole distribution.

Lemma 1. *If the stationary distribution satisfies $\mathbb{E}\left[e^{z}\right] < \infty$, the growth rate in equation (18) coincides with the growth formula (15) in the tail.*

4.3 Perfect Imitation Case

Recall that if the productivity of the matched innovator is z, then the new productivity of the imitator is $z + q$, where q is drawn from an exogenous distribution $G(\cdot)$. To clarify the role of knowledge diffusion, we consider a simple case in which imitation is perfect: $q = 0$. This case relaxes the no-atom assumption on G used in the baseline model.

Lemma 2 (Perfect Imitation Case)**.** *Suppose that the imitation is perfect and deterministic, i.e.,*

$$G\left(q\right) = \begin{cases} 0 & q < 0 \\ 1 & q \geq 0 \end{cases}.$$

Along the BGP, the endogenous stationary distribution has a Pareto tail index

$$\zeta = \frac{\sqrt{2\lambda\left(\alpha\right)\left(1-\delta\right)}}{\sigma}. \tag{19}$$

and the economy grows at rate

$$g = \frac{1}{\eta - 1}\left(\bar{\mu} + \sigma^2\zeta\right), \tag{20}$$

When imitation is perfect and deterministic, the imitator obtains exactly the same productivity as the matched innovator. In the creative-destruction branch, which occurs with probability δ, the imitator replaces the incumbent but inherits the same productivity. Hence, creative destruction generates displacement but does not change the productivity distribution. The distribution changes only when a new variety is born with probability $1 - \delta$, and the effective distribution-changing matching rate is $\lambda(\alpha)(1 - \delta)$. The Pareto tail index ζ is expressed in terms of this matching rate and the variance of own innovation, as shown in previous models in the literature (e.g. Luttmer (2015)). Since the tail density satisfies $f(z) \propto e^{-\zeta z}$, a higher ζ corresponds to a thinner right tail. Thus, as the matching rate $\lambda(\alpha)(1 - \delta)$ increases, the tail becomes thinner. As the variance of own innovation becomes larger, the tail becomes thicker.

The growth rate is expressed as the sum of the drift of active firms in the tail $\bar{\mu}$ and the product of the variance and the Pareto tail index $\sigma^2\zeta$, which is also an analogous result to Luttmer (2015). Following Proposition 2, we can decompose the factors of growth into own innovation and knowledge diffusion:

$$g = \frac{1}{\eta - 1}\left\{\underbrace{\bar{\mu} + \frac{1}{2}\sigma^2\zeta}_{\text{own innovation}} + \underbrace{\frac{1}{2}\sigma^2\zeta}_{\text{knowledge diffusion}}\right\}$$

This growth accounting shows that higher innovation volatility σ contributes to economic growth through both own innovation and knowledge diffusion.

4.4 Appropriability and the BGP

The bargaining weight between innovators and imitators shapes firms' incentives to innovate or imitate. This subsection examines how changes in innovators' appropriability affect the equilibrium composition and growth dynamics of the economy.

As innovators' bargaining weight increases, a larger proportion of firms choose to innovate rather than imitate. Consequently, the diffusion rate $\lambda(\alpha)$ goes down. Stronger innovators' appropriability reduces the speed of knowledge diffusion and, hence, its contribution to aggregate growth.

The innovation intensity is driven by direct and indirect effects from changes in innovators' bargaining weight. Equation (16), which characterizes the upper bound of innovation $\bar{\mu}$, highlights these forces. A higher β lets innovators appropriate a larger share of the rents from knowledge diffusion, which raises the private return to innovation. At the same time, an indirect congestion effect works in the opposite direction: as β increases, α increases and $\lambda(\alpha)$ falls. This negative indirect effect can dominate the positive direct effect. Combining the two forces, the transfer-related return to innovation, summarized by $\beta\lambda(\alpha)$, can be hump-shaped in the innovators' bargaining weight. Finally, the equilibrium growth rate also matters for innovation incentives. A higher equilibrium growth rate raises the effective discounting of firm value in detrended terms, which lowers the incentive to invest in innovation through the denominator in equation (16).

The overall effect of innovators' bargaining weight on the growth rate can therefore be non-monotonic. At low levels of β, higher growth through innovation may offset the decline in growth due to knowledge diffusion.

We further characterize the limiting behavior of the economy as innovators appropriate almost the full rent from knowledge diffusion. As innovators' bargaining weight approaches one, leaving imitators with almost none of the surplus, the equilibrium composition of innovators and imitators adjusts to restore the balance of incentives between innovation and imitation. That is, for the indifference condition (11) to hold, the expected surplus from knowledge diffusion must remain well-defined. This requires the expected surplus to be integrable, i.e., $\zeta > 1$, and the endogenous diffusion rate $\lambda(\alpha)$ to adjust above a minimum level. We summarize this limit below.

Corollary 1 (Limiting Economy)**.** *As innovators' bargaining weight approaches one, $\beta \to 1$, the fraction of innovators $\alpha \to \alpha^1$, which satisfies $\lambda\left(\alpha^1\right) = -\frac{1}{2}\frac{\sigma^2}{D'(1)}$. The tail thickens $\zeta \to 1$, and the innovation upper bound converges to $\bar{\mu} \to \bar{\mu}^1$, where $\bar{\mu}^1$ denotes the corresponding limiting solution to equation* (16)*. The growth rate $g \to \frac{1}{\eta-1}\left(\bar{\mu}^1 + \frac{1}{2}\sigma^2\left(1 - \frac{D(1)}{D'(1)}\right)\right)$.*

Through the lens of the perfect and deterministic imitation example in Lemma 2, $D(1) = 1 - \delta$ and $D'(1) = -(1-\delta)$. Therefore, the growth rate from imitation $\lambda(\alpha)(1-\delta)$ converges to $\frac{1}{2}\sigma^2$, and the overall growth rate converges to $\frac{1}{\eta-1}\left(\bar{\mu}^1 + \sigma^2\right)$.

This limiting case also reveals that the economy exhibits a discontinuity when innovators receive the full bargaining weight. In this extreme case with $\beta = 1$, no firm would want to be on the imitation side, and knowledge diffusion completely disappears, i.e., $\alpha = 1$ and $\lambda(\alpha) = 0$. The growth comes solely from the innovators and reduces to $\frac{1}{\eta-1}\left(\bar{\mu} + \frac{1}{2}\sigma^2\right)$.

Table 3: Calibrated parameters

Parameter	Value	Moment	Data
Standard			
Discount rate ρ	0.032	Annual interest rate	5%
Elasticity of substitution η	4	Preset	–
Innovation process			
Standard deviation σ	0.11	Own innovation randomness (GHK)	0.11
Cost function scale χ	629	Own innovation growth (GHK)	1.15%
Cost function elasticity ε	1	Preset	–
Diffusion process			
Matching function curvature ω	0.5	Preset	–
Matching function efficiency κ	4.8	Total entry rate (GHK)	7.0%
Creative-destruction rate m_{CD}	5.4%	Creative-destruction rate (GHK)	5.4%
New-variety rate m_{NV}	1.6%	New-variety rate (GHK)	1.6%
Quality draw shape ϕ	4.87	Quality draw shape (GHK)	4.87
Quality draw lower bound $\underline{q}$	-1.37	New-variety average quality (GHK)	0.32
Innovator bargaining weight β	0.5	Royalty rate (Kankanhalli Kwan 2022)	–

Notes: The creative-destruction and new-variety entries are annual arrival rates from GHK. The model maps these rates to the primitive branch probability δ and the effective displacement probability $\tilde{\delta} = \delta (1 - G(0))$.

5 Quantitative Evaluation

In this section, we first calibrate the model and then quantitatively assess the impact of innovator appropriability on the overall economic growth.

5.1 Calibration

To calibrate the model, we borrow the estimate from Garcia-Macia et al. (2019) (henceforth GHK) regarding the extent of creative destruction of innovation as opposed to own innovation and the arrival of new varieties based on the patterns of job creation and job destruction at the firm level during the period 1983-2013.[12]

We assume that the economy is on a BGP and take advantage of the properties of the economy along the BGP in Proposition 2 to map the parameters to their corresponding moments. The calibration is carried out at annual frequency: one unit of time in the model corresponds to one year in the data. Table 3 reports the calibrated parameter values. To match an annual interest rate of around 5% and an aggregate consumption growth rate of

[12]The firms in our model are single-product, whereas GHK focus on multi-product firms, so their notion of firms or establishments versus products differs from ours.

1.76%, we set the discount rate ρ to 0.032 according to the interest rate equation $r = \rho + g$. Following the standard in the literature, we set the elasticity of substitution parameter η to 4.

Standard deviation. The standard deviation of the Brownian motion σ corresponds to the volatility of the growth rate of the largest firms in the economy.[13] We instead calibrate the innovation and diffusion process directly from GHK so that expected firm growth due to own innovation satisfies $\bar{\mu} + \frac{1}{2}\sigma^2 = \lambda_i \frac{1}{\phi - 1}$ and the variance of firm growth is $\sigma^2 = \frac{2\lambda_i}{\phi^2}$, where $\mathbb{E}[\lambda_i] = 0.146$.

Innovation cost function. The innovation function is specified as $c(\mu) = \frac{\chi}{\varepsilon+1}\mu^{\varepsilon+1}$. We fix the elasticity parameter ε at 1, following the macro and micro estimates of innovation elasticity around unity in the literature. We calibrate the innovation cost scale parameter χ to match the contribution of own innovation to aggregate growth, $\left(\bar{\mu} + \frac{1}{2}\sigma^2\zeta\right)/(\eta - 1) = 1.15\%$, and obtain $\chi = 629$.

Quality draw. For the quality draw $G(q)$, we specify the exponential distribution in Example 1. First, consider the knowledge diffusion events that lead to creative destruction. The conditional distribution for the quality draw $\tilde{G}(q)$ has the same shape as $G(q)$ after being truncated for $q \geq 0$. This distribution is identical to one specified in Garcia-Macia et al. (2019). Therefore, we map the shape parameter to one they calibrated, $(\eta - 1)\phi = 14.6$, and obtain that $\phi = 4.87$. Next, consider the knowledge diffusion events that lead to new varieties. To match an average quality of new varieties equal to 32% of the incumbents' level, we infer that the lower bound is $\underline{q} = -1.37$.

Creative destruction and new varieties. To gauge the extent of creative destruction, we also borrow the estimates from GHK. They compute that, on an annual basis, the arrival rate of creative destruction is 0.054, and the arrival rate of new varieties is 0.016. Decomposing into entrants and incumbents, the creative destruction rate by entrants is 0.0345 and the rate by incumbents is 0.0195. Our model maps to the GHK environment except for one discrepancy: in contrast to their multi-product firms, the firms in our model are single-product firms and an incumbent does not take over a variety owned by another firm. Thus, creative destruction occurs only through entrants. For the purpose of our

[13] Using the Longitudinal Business Database, Davis, Haltiwanger, Jarmin and Miranda (2007) find that the volatility of the growth rate for large firms is in the range of 0.05 to 0.1. This measure of the volatility of firm growth rate in Davis et al. (2007) excludes short-lived firms and entry and exit.

analysis, we lump creative destruction by incumbents into creative destruction by entrants to capture sources of knowledge diffusion.[14]

Let $m_{CD} = 0.054$ denote the annual arrival rate of creative destruction and $m_{NV} = 0.016$ the annual arrival rate of new varieties. These are rate targets, not primitive probabilities. In the model they satisfy

$$m_{CD} = \lambda(\alpha)\,\delta\,(1 - G(0)), \qquad m_{NV} = \lambda(\alpha)\,(1 - \delta).$$

Given the calibrated quality draw distribution, the share of creative-destruction draws that can displace the incumbent is $1 - G(0) = 0.0013$. The two rate restrictions imply $\delta = 0.9996$ and $\tilde{\delta} \equiv \delta(1 - G(0)) = 0.00127$. The ratio $s_{CD} = m_{CD}/(m_{CD} + m_{NV}) = 0.77$ summarizes the GHK decomposition of observed diffusion events, but it is not a primitive model probability.

Bargaining weight. The innovator's bargaining weight is a parameter that we know little about. Several recent empirical studies help fill this gap. Using licensing transactions for US public firms, Kankanhalli and Kwan (Forthcoming) finds that the royalty rate on average amounts to 11.5% of net sales, or around a half-and-half split of the profits. Therefore, we set β to 0.5, assigning a half-and-half split of entry surplus to the transacting parties.[15]

To gauge whether this calibrated bargaining weight is sensible, we cross-check a few model moments directly related to the bargaining weights. First, the bargaining weight affects a firm's indirect payoff from innovation in addition to direct profit from production. Our calibrated model implies that licensing contributes to around 12.8% of total revenue for the largest firms.[16] Second, the bargaining weight β shapes the private return of innovation relative to the social return. When the innovators appropriate all gains from knowledge diffusion, the gap between the private return and the social return closes to zero. Bloom et al. (2013) estimate that the social rate of return to R&D is around 55%, while the private return is 21%. While our model doesn't directly map to their calculation, we have a notion of social value versus private value, which has a ratio of 2.5.[17]

[14] Another interpretation of our results is imitation at the product level, and firms are a collection of products.

[15] Practitioners in licensing transactions commonly refer to a "25% rule", which states that the licensees pay a 25% royalty rate out of the profits from the licensing deals to the licensors.

[16] For the largest firms, the licensing income is $\beta\lambda(\alpha)s^e v$, and the direct revenue is $\eta\pi(1 - c(\bar{\mu}))$. The share $\frac{\beta\lambda(\alpha)s^e v}{\eta\pi(1-c(\bar{\mu}))+\beta\lambda(\alpha)s^e v} = 12.8\%$.

[17] The private value of R&D is the right-hand side of equation (16), while the social value is obtained by setting $\beta = 1$ in the same expression.

Matching function. We work with the Cobb-Douglas matching function: $M(\alpha, 1-\alpha) = \kappa\alpha^{\omega}(1-\alpha)^{1-\omega}$. The corresponding contact rate for innovators is $\lambda(\alpha) = \kappa\left(\frac{\alpha}{1-\alpha}\right)^{\omega-1}$ and for imitators $\iota(\alpha) = \kappa\left(\frac{\alpha}{1-\alpha}\right)^{\omega}$. The level parameter κ captures the ease of knowledge diffusion. The matching elasticity $\omega \in [0,1]$ controls the contribution of innovators to matching.

Although this specification of the matching function is widely examined in studies of labor markets, it is less explored in studies related to ours for technology markets. One exception is Akcigit, Celik and Greenwood (2016) who look into the patent resale market through a random search model. They also specify a Cobb-Douglas matching function and, using the empirical distribution of the duration until a patent gets sold, find an equal amount of congestion on the two sides of the market. Hence, we set ω to 0.5 such that the extent of congestion is equal on the two sides. The parameter κ is calibrated to match the contribution of knowledge diffusion to growth and the observed entry rates. Given δ, this is equivalent to choosing the equilibrium contact rate so that $\lambda(\alpha)\delta(1-G(0)) = m_{CD}$ and $\lambda(\alpha)(1-\delta) = m_{NV}$. The resulting κ is reported in Table 3. While the magnitude of the model-implied innovator share is not directly interpretable, it is helpful to view the pool of imitators as a large pool of potential entrants.

5.2 Growth Decomposition

We decompose the growth rate into different sources using the calibrated model. We do so both for firms in the tail according to equation (15) and for firms across the whole distribution according to equation (18).

Overall, the growth decomposition in the tail is more or less identical to the one across the whole distribution. Own innovation accounts for 2/3 of overall growth, and net knowledge diffusion accounts for the remaining 1/3. The two decompositions indeed sum to the same total growth rate, as stated in Lemma 1. The small discrepancies in the decomposed levels stem from cross-sectional heterogeneity in the calibrated model. In particular, the growth due to own innovation in the tail is always higher than the level that averages over the entire distribution, i.e., $\bar{\mu} + \frac{1}{2}\sigma^2\zeta > \mathbb{E}\left[\mu(z)e^z\right]/\mathbb{E}\left[e^z\right] + \frac{1}{2}\sigma^2$. The gap remains modest for two reasons. First, the Pareto tail index ζ implied by equation (17) is 1.17, close to the empirical benchmark of about 1.1 in the U.S. data. Second, cross-sectional variation in innovation intensity is limited. Consistent with this observation, the margin-to-mean ratio, $e^{\underline{z}}/\mathbb{E}\left[e^z\right]$, is only 0.019, indicating that firms exiting at the activity margin are small relative to the average incumbent and that the exit adjustment in equation (18) is quantitatively small.

Counterfactuals: Appropriability and Growth We evaluate how innovator appropriability in knowledge diffusion affects the incentives for innovation and imitation and hence the overall economic growth. We use the calibrated model to carry out the counterfactual exercise by varying the bargaining weight. Figure 1 plots, as the innovator's bargaining weight β increases from 0 to 1, the corresponding growth rate and its decomposition into innovation and knowledge diffusion.

The growth rate is maximized when the innovator's bargaining weight is 0.86. The optimal bargaining weight trades off two opposing forces for growth. On the one hand, it discourages knowledge diffusion as imitation becomes less profitable. On the other hand, it encourages innovation, as the innovators appropriate a larger share of the surplus created by knowledge transfer.[18] We find that the first effect is very mild since, as β is increased from 0 to 0.86, the fraction of imitators falls very slowly from 58% to 47%. This in turn explains why the contribution of knowledge diffusion to growth depicted in Figure 1 is so flat as a function of β. On the other hand, the elasticity of innovation to the innovator's bargaining weight is large, as the innovator's return from licensing represents 20% of their total revenue at the calibrated level and it varies quite substantially with the innovator's appropriability. The indirect payoff from licensing increases from 0% to 47% of total revenue as β is increased from 0 to 0.86. This explains the fairly steep rise in the contribution of innovation to growth observed in Figure 1. Overall, the long-run growth rate increases from the benchmark value of about 2% at the bargaining weight $\beta = 0.5$ to 2.31% for the optimal value $\beta = 0.86$. It should be noted that while this optimal bargaining weight might seem high, the net share of the value appropriated by innovators, $\left(\beta^* s^e - \tilde{\delta}\right) / \left(s^e - \tilde{\delta}\right)$, is much lower, representing 0.58 of the net surplus created.

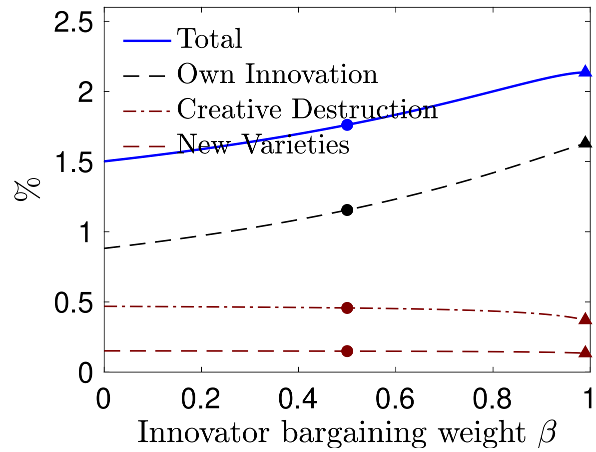
5.3 Ramsey Problem

In this section, we analyze the optimal bargaining weight in our dynamic model. A Ramsey planner chooses the stream of bargaining weights $\left\{\beta\left(t\right)\right\}_{t\geq 0}$. By doing so, she influences firms' choices so as to maximize household utility. We focus on the Ramsey steady state that emerges as a result of the stream of optimal bargaining weights.

For tractability, we treat the innovation intensity μ as exogenous and, following Section 4.3, assume imitators perfectly replicate the incumbents' productivity so that $q = 0$ at each diffusion event. Under this assumption, we can characterize the optimal bargaining weight analytically. Once the share of innovators α in the steady-state equilibrium is determined, the steady state is uniquely pinned down. Therefore, we follow the steps

[18] At very high levels of β there is a Laffer curve as the share of imitators vanishes.

Figure 1: Bargaining weight and growth rate



Notes: Growth rate and its innovation and diffusion components as the innovator's bargaining weight β varies from 0 to 1. Growth peaks at $\beta = 0.86$, where the rate reaches 2.31%.

below to obtain the optimal bargaining weight β in the Ramsey steady state. (i) First, we solve the benevolent social planner's problem, which maximizes household utility subject to the KFE (13), and compute the steady-state value of α. (ii) Next, we identify the value of β that generates this α in the competitive steady state.

The following proposition characterizes the Ramsey steady state when the innovation intensity μ is exogenous.

Proposition 3 (Ramsey Steady State with Exogenous Innovation)**.** *When the innovation intensity μ is exogenous and imitators perfectly replicate the incumbents' productivity ($q = 0$), the optimal bargaining weight β and the share of innovators α in the Ramsey steady state are characterized by the following equations:*

$$\left(\rho - \lambda\left(\alpha\right)\left(1-\delta\right)\right)\alpha\left(1+\frac{1}{\tilde{\nu}}\right) = \psi\left(\alpha\right)\left(1-\delta\right)\left(1-\omega\right)\left(\left(\frac{\zeta}{\zeta-1}\right)^2 - 1 + \frac{1}{\tilde{\nu}}\left(\left(\frac{\zeta}{\zeta+\tilde{\nu}}\right)^2 - 1\right)\right), \tag{21}$$

$$\left(\rho - \lambda\left(\alpha\right)\left(1-\delta\right)\beta\right)\left(1+\frac{1}{\nu}\right) = \psi\left(\alpha\right)\left(1-\delta\right)\left(1-\beta\right)\left(\left(\frac{\zeta}{\zeta-1}\right)^2 - 1 + \frac{1}{\nu}\left(\left(\frac{\zeta}{\zeta+\nu}\right)^2 - 1\right)\right), \tag{22}$$

where ζ is pinned down by (19). The term $\tilde{\nu}$ in the planner's problem equation (21) and the term

ν in the competitive equilibrium equation (22) satisfy

$$\left(\frac{\zeta}{\zeta+\tilde{\nu}}\right)^2 = \frac{\lambda(\alpha)(1-\delta)}{\rho},$$
$$\left(\frac{\zeta}{\zeta+\nu}\right)^2 = \frac{\lambda(\alpha)(1-\delta)}{\rho+\lambda(\alpha)(1-\delta)\left(1-\beta\right)}.$$

Under exogenous μ, the steady-state share of innovators α in the benevolent planner's allocation of the dynamic model is characterized by equation (21). The relationship between the bargaining weight β and the share of innovators α in the competitive steady state is given by equation (22). Therefore, substituting the α obtained from (21) into (22) yields the optimal bargaining weight β.

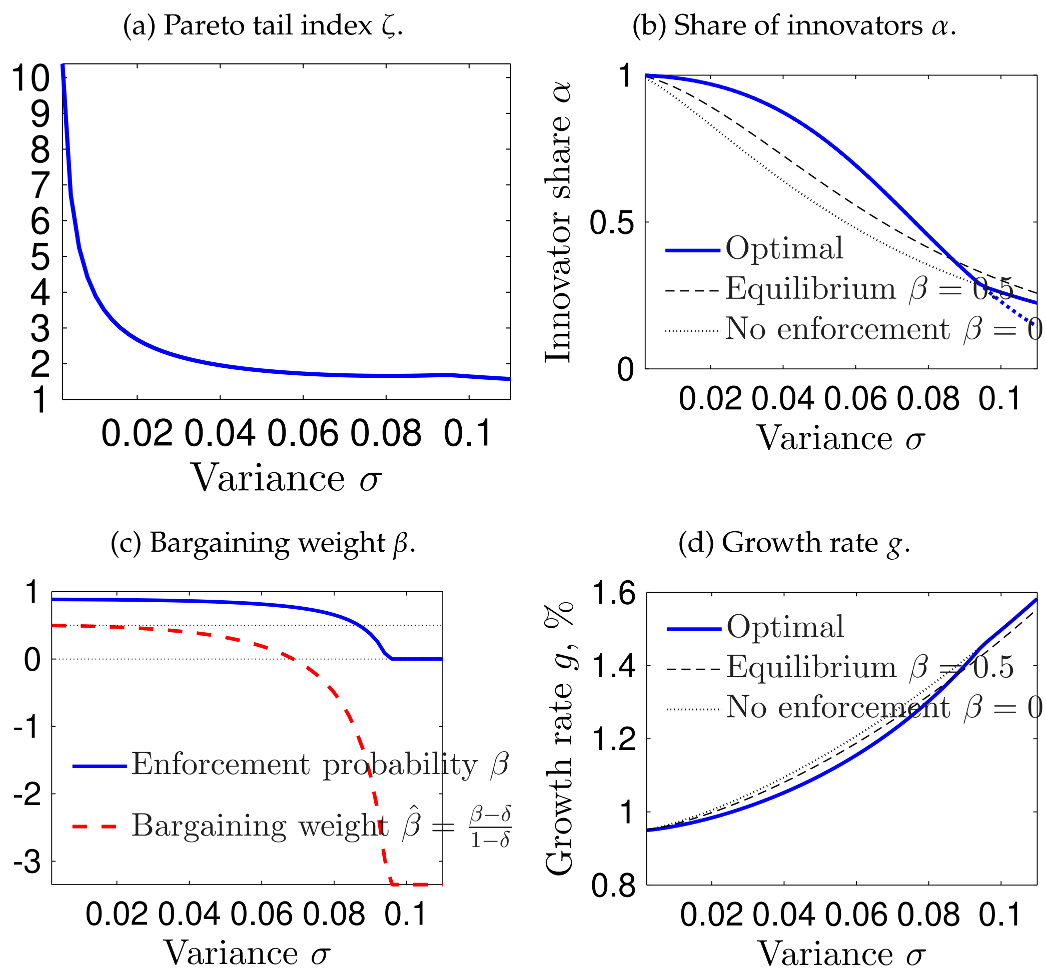
In implementing this Ramsey problem with an exogenous innovation intensity, we retain the parameter values from the dynamic calibration in Section 5.1. The exogenous innovation intensity is set to $\mu = 0.0285$ so that the own-innovation contribution to aggregate growth, $\left(\mu + \frac{1}{2}\sigma^2\zeta\right)/\left(\eta-1\right)$, matches the 1.15% estimate reported by Garcia-Macia et al. (2019) at the benchmark.

As shown in Figure 2a, higher innovation volatility σ generates greater productivity heterogeneity and lowers the Pareto tail index ζ. A thicker tail makes imitation appealing to more firms, so the share of innovators α drops (Figure 2b). When heterogeneity is large, the displacement effect is strong, as discussed in Section 2.2, and the planner assigns innovators a lower bargaining weight, reducing β (Figure 2c).

When σ is high, the optimal bargaining weight falls below the baseline that assumes $\beta = 0.5$ (Figure 2c). In that range, the share of innovators falls under the baseline (Figure 2b). Faster knowledge diffusion in the Ramsey steady state then raises the growth rate above the benchmark outcome (Figure 2d).

Figure 3 shows the optimal bargaining weight for different matching elasticities ω. As in Section 2.2, a higher matching elasticity ω magnifies the congestion externality on the imitator side, so the planner assigns innovators a higher bargaining weight and raises β. In the dynamic model, the Hosios (1990) condition $\beta = \omega$ does not hold exactly, and the gap widens as the productivity heterogeneity induced by σ increases.

Figure 2: Planner allocations under the Ramsey steady state



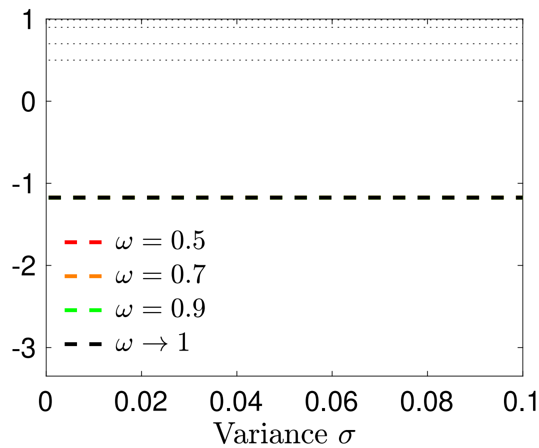
Notes: Panels display the planner solutions for the Pareto tail index, the share of innovators, the bargaining weight, and the growth rate, respectively. The x-axis reports innovation volatility, σ, so each panel compares the allocation under alternative values of the Brownian standard deviation.

6 Extensions

6.1 Scalable-Volatility Innovation Process

The counterfactual analysis above reveals that the endogenous level of innovator heterogeneity is critical for determining the optimal bargaining weight. One important factor driving this heterogeneity is the variance of innovation outcomes. In the model in Section 3, regardless of innovation intensity, the variance of the outcome is fixed. We call this a *fixed-volatility* productivity process, represented by a Brownian motion with an upward drift μ and a standard deviation σ. This model of innovation is commonly used in the

Figure 3: Bargaining weights across matching elasticity



Notes: The figure plots the Ramsey planner's optimal bargaining weight $\hat{\beta}$ as the curvature of the matching function, ω, varies. The x-axis reports σ, and the lines compare allocations across alternative levels of matching elasticity, ω.

literature, for example, in the continuous-time model of Luttmer (2007).

We contemplate an alternative setting in which a firm's endogenous innovation intensity can affect not only the expected level but also the variance of productivity improvement. We capture the effect on the variance with a flexible specification of the Brownian motion, which has a standard deviation $\sigma(\mu)$ as a function of innovation intensity μ. Specifically, we introduce a *scalable-volatility* innovation process, represented by a Brownian motion with standard deviation $\sigma(\mu) = \sigma\sqrt{\mu}$. That is, firm productivity follows

$$dz = \mu dt + \sigma\sqrt{\mu}dB.$$

To micro-found this scalable-volatility productivity process, we consider firm innovation as experimentation through a set of risky projects. Let μ denote the number of projects carried out at a given point in time. Each project generates some incremental productivity improvement with an outcome drawn independently from a normal distribution $N(1, \sigma^2)$. Together the μ projects generate productivity improvement drawn from a normal distribution $N(\mu, \sigma^2\mu)$. Firm innovation follows a quality ladder in which each discovery raises productivity by one rung.

A distinctive feature of the scalable-volatility productivity process is that there is no free luck in innovative discoveries. If firms expend more resources in innovation and carry out more experimentation, they are more likely to be lucky in discovering better productive processes, leading to more variable innovation outcomes. The level of innovation intensity affects the variance of innovation outcomes and in turn the equilibrium extent of firm heterogeneity.

We calibrate the dynamic model with the scalable-volatility productivity process, following the same strategy as with the fixed-volatility model. Only three parameters are altered here. First, the growth decomposition into innovation and knowledge diffusion is unchanged. After obtaining innovation intensity from the decomposition, to fit the innovator's innovation volatility $\sigma\sqrt{\bar{\mu}}$ to 0.08, we back out σ as 0.94. Second, since the optimality condition for innovation is altered now to account for the additional effect of higher innovation volatility on firm value, to target the innovation intensity, we set the innovation-cost parameter γ to 4100. Third, to target the knowledge diffusion rate, we set ϕ to 0.038. The model-implied share of innovators is 49%.

We then carry out the counterfactual exercise for the growth-maximizing innovator bargaining weight. We trace the corresponding growth rate and its decomposition into innovation and diffusion as the innovator's bargaining weight β increases from 0 to 1. Compared to the fixed-volatility model, here the overall growth rate tends to be maximized by an even higher innovator bargaining weight, $\beta^* = 0.9$. The corresponding net share appropriated by innovators, $\left(\beta^* - \delta\right)/(1 - \delta)$, is 0.7. The scope of improving long-run growth is larger with scalable volatility: the growth rate increases from the calibrated value of 2% at the benchmark bargaining weight $\beta = 0.5$ to 2.66% at the optimal value $\beta^* = 0.9$.

One distinctive feature of the scalable-volatility innovation process is how knowledge diffusion responds to innovator appropriability. With fixed volatility, growth through knowledge diffusion decreases monotonically as the innovator's bargaining weight increases. In contrast, with scalable volatility, for a large interval of the innovator's bargaining weight, the diffusion rate increases as the innovator's bargaining weight increases. This is because, at a higher innovator bargaining weight, as innovation intensity $\bar{\mu}$ increases, innovation volatility $\sigma\sqrt{\mu}$ also increases. This distinction can also be seen through the pattern of the Pareto tail index, which experiences a sharper decrease in the case with scalable volatility compared to the case with fixed volatility. As a result, the decline in the share of imitators is even milder: it falls from 49% to 44% as β increases from 0 to 0.86. The response of innovation intensity, on the other hand, is similar in the two cases. This is because the indirect payoff to innovation is largely driven by the change in appropriability in both cases. Here, the innovator's return from licensing represents 19% of their total revenue at the calibrated level and it increases from 0% to 49% of total revenue as β is increased from 0 to 0.9.

7 Conclusion

Recent macro models of growth and diffusion, such as Moll and Lucas (2014) and Perla and Tonetti (2014), highlight how random matching spreads ideas. Our contribution is to bring three missing ingredients to that framework and quantify their interaction. First, we model matching frictions explicitly, allowing the ratio of innovators to imitators to shape contact rates. Second, we incorporate creative destruction so that imitators can displace innovators and erode their rents. Third, we let the bargaining weights of innovators and imitators span the full range of surplus splits, tying innovator appropriability to measured growth outcomes.

The static model delivers a modified Hosios condition for homogeneous firms, clarifying how congestion forces adjust the benchmark share innovators should appropriate. Once we allow for heterogeneity, two forces push the optimum away from that benchmark: innovators earn additional rents from heterogeneous productivity draws, and the marginal match becomes less valuable to imitators. Consequently, the optimal bargaining weight declines with productivity heterogeneity and can fall to zero when heterogeneity is sufficiently pronounced.

These insights hold in our quantitative dynamic model. The calibrated model implies that growth is maximized at an innovator bargaining weight of $\beta^* = 0.86$. This raises the long-run growth rate from 2.00 to 2.31 percent. Net of creative destruction, this optimum corresponds to innovators appropriating about 58 percent of the surplus created through licensing. When we vary the matching elasticity, the optimal β that maximizes social welfare spans the full range from 0 to 1, underscoring how congestion in the matching market is pivotal for policy.

At the current stage, the dynamic environment keeps innovation intensity exogenous while we study optimal bargaining weights. Future work will endogenize the innovation intensity and solve the resulting Ramsey problem. This will let us trace welfare implications and provide further guidance for intellectual property design.

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Appendix

A Derivations and Proofs

A.1 Proof of Proposition 1

Proof. Let $\bar{z} \equiv \mathbb{E}\left[z | z \geq \underline{z}\right]$ and write $M = M(\alpha, 1-\alpha)$. Along the equilibrium locus defined by the marginal firm's indifference condition, write $\beta = B(\alpha)$ and $\mu = \mu(\alpha)$. Since $\alpha = 1 - F\left(\underline{z}\right)$,

$$\frac{d}{d\alpha}(\alpha\bar{z}) = \underline{z}, \qquad \frac{d\bar{z}}{d\alpha} = \frac{\underline{z} - \bar{z}}{\alpha}.$$

Let M' denote the total derivative of $M(\alpha, 1-\alpha)$ with respect to α. Constant returns to scale imply

$$\frac{M'}{M} = \frac{\omega}{\alpha} - \frac{1-\omega}{1-\alpha},$$

where ω is the elasticity of the matching function with respect to its innovator input.

The planner's objective is

$$\mathcal{W} = \alpha\bar{z}\left(\mu - c\left(\mu\right)\right) + M\bar{z}\mu.$$

Differentiating along the equilibrium locus gives

$$\frac{d\mathcal{W}}{d\alpha} = \underline{z}\left(\mu - c\left(\mu\right)\right) + \mu\left(M'\bar{z} + M\frac{\underline{z} - \bar{z}}{\alpha}\right) + M\bar{z}\left(1-\beta\right)\mu',$$

where the coefficient on μ' uses the innovation first-order condition $c'\left(\mu\right) = 1 + \beta M/\alpha$:

$$\alpha\bar{z}\left(1 - c'\left(\mu\right)\right) + M\bar{z} = M\bar{z}\left(1-\beta\right).$$

The marginal firm's indifference condition can be rewritten as

$$\underline{z}\left(\mu - c\left(\mu\right)\right) = M\mu\left[\frac{\left(1-\beta\right)\bar{z}}{1-\alpha} - \frac{\beta\underline{z}}{\alpha}\right].$$

Substituting this expression and the derivative of the matching function into $d\mathcal{W}/d\alpha = 0$, and then dividing by $M\mu/\alpha$, yields

$$0 = \left(1-\beta\right)\underline{z} - \left(1-\omega\right)\bar{z} + \frac{\alpha}{1-\alpha}\left(\omega - \beta\right)\bar{z} + \left(1-\beta\right)\bar{z}\frac{\alpha\mu'}{\mu}.$$

Regrouping terms gives

$$(\omega-\beta)\left(\underline{z}+\frac{\alpha}{1-\alpha}\bar{z}\right)-(1-\omega)\left(\bar{z}-\underline{z}\right)+\left(1-\beta\right)\bar{z}\frac{\alpha\mu'}{\mu}=0.$$

Finally,

$$\mathcal{E}_{\mu,\alpha}\equiv\frac{d\log\mu\left(\alpha\right)}{d\log\alpha}=\frac{\alpha\mu'}{\mu},$$

which delivers equation (4).

This elasticity is a total elasticity along the equilibrium locus. To see this explicitly, differentiating

$$c'\left(\mu\left(\alpha\right)\right)=1+B\left(\alpha\right)\lambda\left(\alpha\right)$$

implies

$$\mu'=\frac{1}{c''\left(\mu\right)}\left[B'\left(\alpha\right)\lambda\left(\alpha\right)+B\left(\alpha\right)\lambda'\left(\alpha\right)\right].$$

Thus $\mathcal{E}_{\mu,\alpha}$ includes both the direct effect of changing appropriability and the equilibrium contact-rate effect induced by the change in α. □

A.2 The Extended Enforcement Game

In the one-period model presented in Section 2, we assumed for illustrative purposes that knowledge transfers always create new varieties, so the enforcement game that determines the surplus division is straightforward. In the dynamic model in Section 3, by contrast, a transfer may lead to creative destruction, allowing the imitator to displace the incumbent. This changes the outside options of both parties in the licensing stage.

For simplicity, we assume that the imitation outcome is realized before the enforcement game.[19] Figure 4 depicts the agents' payoffs in the cases in which entry occurs. When the realized draw does not permit entry, the match leaves both parties at their status quo payoffs.

Condition on a meeting between an innovator with productivity z and an imitator at time t, and let the realized imitation outcome be q. Define

$$S\left(z,t\right)\equiv V\left(z,t\right)-W\left(t\right).$$

After the realization of q, the imitator's entry surplus is therefore $S\left(z+q,t\right)$. Let t_L denote

[19] If the agents can commit to licensing agreements, the results remain unchanged even if they play the enforcement game before the realization of imitation outcomes.

the license fee. The imitator accepts whenever

$$W(t) + S\left(z+q,t\right) - t_L \geq W(t) + \left(1-\beta\right) S\left(z+q,t\right),$$

which is equivalent to $t_L \leq \beta S\left(z+q,t\right)$. Since the innovator's payoff under licensing is increasing in t_L, it sets $t_L = \beta S\left(z+q,t\right)$. When the imitator is indifferent, we assume that it accepts the license.

If $z+q < \underline{z}(t)$ in the new-variety case, or if $q < 0$ in the creative-destruction case, entry does not occur and no license is agreed. We therefore focus on the states in which entry is feasible.

If the transfer leads to a new variety and the imitator enters, i.e., $z+q \geq \underline{z}(t)$, the innovator remains active regardless of the licensing decision. With a license, the innovator obtains $V(z,t) + \beta S\left(z+q,t\right)$ and the imitator obtains $W(t) + \left(1-\beta\right) S\left(z+q,t\right)$. Without a license, the imitator uses the transferred knowledge without authorization and enters anyway; with probability β, enforcement excludes it from production and it receives $W(t)$. Hence its expected payoff is again $W(t) + \left(1-\beta\right) S\left(z+q,t\right)$, so the offer is accepted.

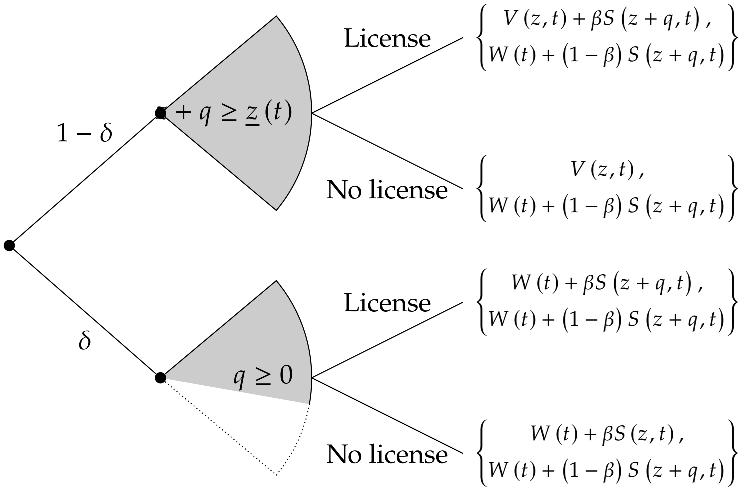
If the transfer leads to creative destruction, the imitator can replace the incumbent only when $q \geq 0$. With a license, the incumbent is displaced and receives $W(t) + \beta S\left(z+q,t\right)$, while the imitator receives $W(t) + \left(1-\beta\right) S\left(z+q,t\right)$. Without a license, the imitator again obtains $W(t) + \left(1-\beta\right) S\left(z+q,t\right)$, whereas the innovator keeps its position with probability β and is displaced with probability $1-\beta$. Its expected payoff is therefore $W(t) + \beta S(z,t)$. The innovator weakly prefers licensing whenever $S\left(z+q,t\right) \geq S(z,t)$, which holds for $q \geq 0$; the participation constraint is binding when $q = 0$ and strict when $q > 0$, so the license is accepted for all $q \geq 0$.

Aggregating over the imitation outcomes that lead to entry, the innovator's expected indirect net contribution from a match is

$$\begin{aligned}
&\delta \int_{q\geq 0} \left[\beta S\left(z+q,t\right) - S(z,t)\right] dG\left(q\right) \\
&\quad + (1-\delta) \int_{z+q\geq \underline{z}(t)} \beta S\left(z+q,t\right) dG\left(q\right) \\
&= \beta S^e(z,t) - \tilde{\delta} S(z,t),
\end{aligned}$$

where the first term captures creative-destruction matches and the second term captures new-variety matches. This is exactly the term that appears in equation (8). The imitator

Figure 4: Events and agent payoffs in the extended enforcement game



retains the complementary payoff

$$\left(1-\beta\right) S^{e}\left(z,t\right),$$

as in equation (9). If we express the innovator's payoff as a state-dependent share of the total surplus generated by a meeting, that total surplus can be written as

$$S^{e}\left(z,t\right)-\tilde{\delta}S\left(z,t\right).$$

Therefore, the corresponding implicit innovator bargaining weight is

$$\hat{\beta}\left(z,t\right)=\frac{\beta S^{e}\left(z,t\right)-\tilde{\delta}S\left(z,t\right)}{S^{e}\left(z,t\right)-\tilde{\delta}S\left(z,t\right)}.$$

Under perfect imitation, $q=0$ almost surely. Therefore,

$$S^{e}\left(z,t\right)=S\left(z,t\right)\quad\text{and}\quad\tilde{\delta}=\delta.$$

Hence the implicit bargaining weight reduces to

$$\hat{\beta}\left(z,t\right)=\frac{\beta-\delta}{1-\delta}.$$

A.3 Aggregation

Consider the problem of a representative final good producer that uses the intermediate goods as inputs. Let $P(t)$ denote the price of the final good and $p(z,t)$ the price of the intermediate good produced by a firm with productivity z. Given the aggregate production function in equation (5), the final good producer chooses the intermediate goods:

$$\max_{y(z,t)} P(t) Y(t) - \int p(z,t)\, y(z,t)\, dF(z,t).$$

The solution yields the demand for the intermediate goods:

$$\frac{y(z,t)}{Y(t)} = \left(\frac{p(z,t)}{P(t)}\right)^{-\eta}. \tag{23}$$

The zero-profit condition for the final good producer implies that the aggregate price is

$$P(t) = \left[\int p(z,t)^{1-\eta}\, dF(z,t)\right]^{\frac{1}{1-\eta}}. \tag{24}$$

Next, consider the profit maximization of a firm with productivity z. Given the innovation decision rule $\mu(z,t)$, the firm chooses price and labor input,

$$\pi(z,t) = \max_{p,\ell} p\left(\left(1 - c\left(\mu(z,t)\right)\right) e^{z}\right)^{\frac{1}{\eta-1}} \ell - w(t)\,\ell$$

subject to the demand function in equation (23). The firm charges a markup $\frac{\eta}{\eta-1}$ over the unit cost:

$$p(z,t) = \frac{\eta}{\eta-1}\left(\left(1 - c\left(\mu(z,t)\right)\right) e^{z}\right)^{\frac{1}{1-\eta}} w(t). \tag{25}$$

Substituting equation (25) into equation (24), we obtain that

$$P(t) = \frac{\eta}{\eta-1}\frac{w(t)}{Z(t)^{\frac{1}{\eta-1}}},$$

which is normalized to 1, implying that $w(t) = \frac{\eta-1}{\eta} Z(t)^{\frac{1}{\eta-1}}$. The firm's labor input is

$$\ell(z,t) = \left(1 - c\left(\mu(z,t)\right)\right) e^{z} Z(t)^{-\frac{\eta}{\eta-1}} Y(t).$$

The labor market clearing condition, $\int \ell(z,t)\, dF(z,t) = L$, implies that

$$Y(t) = Z(t)^{\frac{1}{\eta-1}} L.$$

The firm's maximized profit from production is

$$\pi(z,t) = \Pi(t)\left(1 - c\left(\mu(z,t)\right)\right) e^{z}, \text{ where } \Pi(t) = \frac{1}{\eta} Z(t)^{\frac{2-\eta}{\eta-1}} L.$$

A.4 Innovation Pattern

We formally establish the cross-sectional pattern of firm innovation in the following lemma.

Lemma 3 (Innovation Pattern). *The innovation intensity $\mu(z,t)$ increases in the level of productivity z and converges to an upper bound:*

$$\lim_{z\to\infty} \mu(z,t) = \bar{\mu}(t).$$

Proof. Divide the entry surplus equation (7) by e^{z}:

$$\frac{S^{e}(z,t)}{e^{z}} = \delta \int_{q\geq 0} \left(\frac{V(z+q,t)}{e^{z}} - \frac{W(t)}{e^{z}}\right) dG(q) + (1-\delta) \int_{z+q\geq \underline{z}(t)} \left(\frac{V(z+q,t)}{e^{z}} - \frac{W(t)}{e^{z}}\right) dG(q). \tag{26}$$

Similarly, divide the HJB equation (8) by e^{z}:

$$\begin{aligned} r(t)\frac{V(z,t)}{e^{z}} = \max_{\mu}\Bigg\{ & \Pi(t)\left(1-c\left(\mu\right)\right) + \mu \frac{V_{z}(z,t)}{e^{z}} + \frac{1}{2}\sigma^{2}\frac{V_{zz}(z,t)}{e^{z}} + \frac{V_{t}(z,t)}{e^{z}} \\ & + \lambda(\alpha(t))\left[\beta\frac{S^{e}(z,t)}{e^{z}} - \tilde{\delta}\left(\frac{V(z,t)}{e^{z}} - \frac{W(t)}{e^{z}}\right)\right]\Bigg\}. \end{aligned} \tag{27}$$

We guess and verify that the innovator's value function is asymptotically affine in e^{z}:

$$\lim_{z\to\infty} \frac{V(z,t)}{e^{z}} = v(t). \tag{28}$$

In equation (26) , on the right-hand side, the component due to imitator's outside option relative to the innovator productivity $\frac{W(t)}{e^{z}}$ is decreasing in z. In particular, for firms on the knowledge frontier, this component converges to zero, $\lim_{z\to\infty} \frac{W(t)}{e^{z}} = 0$. Therefore,

$$\lim_{z\to\infty} \frac{S^{e}(z,t)}{e^{z}} = s^{e} v(t), \tag{29}$$

where $s^e = \delta \int_{q \geq 0} e^q dG(q) + (1-\delta) \int e^q dG(q)$.

Differentiating equation (28) with respect to z and t, we obtain that

$$\lim_{z\to\infty} \frac{V_z(z,t)}{e^z} = \lim_{z\to\infty} \frac{V_{zz}(z,t)}{e^z} = v(t) \text{ and } \lim_{z\to\infty} \frac{V_t(z,t)}{e^z} = v_t(t). \tag{30}$$

Substituting the expressions in (29), (28) and (30) into equations (26) and (27), we obtain an ordinary differential equation that characterizes $v(t)$:

$$r(t)\, v(t) = \max_{\mu} \left\{ \Pi(t) \left(1 - c\left(\mu\right)\right) + \left(\mu + \frac{1}{2}\sigma^2 + \lambda\left(\alpha(t)\right)\left(\beta s^e - \tilde{\delta}\right)\right) v(t) + v_t(t) \right\}.$$

The first-order condition with respect to μ at the limit is:

$$\Pi(t)\, c'\left(\bar{\mu}(t)\right) = v(t).$$

Thus we have verified that $V(z,t)$ is asymptotically affine in e^z.

We take the limit of the first-order condition (10):

$$\lim_{z\to\infty} \Pi(t)\, c'\left(\mu(z,t)\right) = v(t).$$

It must be that $\lim_{z\to\infty} \mu(z,t) = \bar{\mu}(t)$. □

A.5 Proof of Proposition 2

Endogenous innovation intensity. Consider a BGP with a growth rate g. The interest rate is $r(t) = r = \rho + g$. The fraction of innovators is $\alpha(t) = \alpha$. The surplus function satisfies $S\left(z + \left(\eta - 1\right) gt, t\right) = e^{gt} S(z)$. The value functions satisfy $V\left(z + \left(\eta - 1\right) gt, t\right) = e^{gt} V(z)$ and $W(t) = e^{gt} W$. The activity threshold grows at rate $\left(\eta - 1\right) g$: $\underline{z}(t) = \underline{z} + \left(\eta - 1\right) gt$. The aggregate productivity grows at rate $\left(\eta - 1\right) g$: $Z(t) = e^{\left(\eta - 1\right) gt} Z$. The profit function satisfies $\Pi(t) = e^{\left(2 - \eta\right) gt} \pi$, where $\pi = \frac{1}{\eta} Z^{\frac{1}{\eta - 1} - 1} L$. Applying these properties, the entry surplus equation (7) and the HJB equations (8) and (9) become

$$S^e(z) = \delta \int_{q\geq 0} (V(z+q) - W)\, dG(q) + (1-\delta) \int_{z+q\geq \underline{z}} (V(z+q) - W)\, dG(q) \tag{31}$$

$$\rho V(z) = \max_{\mu} \Big\{ \pi (1 - c(\mu)) e^z + (\mu - (\eta - 1) g) V_z(z) + \frac{1}{2}\sigma^2 V_{zz}(z) \tag{32}$$
$$+ \lambda(\alpha) \left[\beta S^e(z) - \tilde{\delta}(V(z) - W)\right] \Big\}$$

$$\rho W = \iota(\alpha)(1-\beta)\, \mathbb{E}[S^e(z)].$$

The value matching and smooth pasting conditions (11) and (12) become

$$V(\underline{z}) = W \text{ and } V_z(\underline{z}) = 0.$$

To solve for the endogenous innovation decision, we apply the result in Lemma 3. According to equations (31) and (32), the functions $S^e(z)$ and $V(z)$ on the knowledge frontier is asymptotically affine in e^z:

$$\lim_{z\to\infty} \frac{S^e(z)}{e^z} = s^e v \text{ and } \lim_{z\to\infty} \frac{V(z)}{e^z} = v,$$

where

$$v = \pi \frac{1 - c(\bar{\mu})}{\rho + (\eta - 1) g - \bar{\mu} - \frac{1}{2}\sigma^2 - \lambda(\alpha)(\beta s^e - \tilde{\delta})}.$$

Combining the expression for v above with the first-order condition (10) for innovation, we obtain equation (16) for characterizing the innovation upper bound $\bar{\mu}$.

Endogenous growth rate and distribution. Detrending the productivity distribution according to equation (14), we transform the KFE (13) into an ordinary differential equation: $\forall z \geq \underline{z}$,

$$\lambda(\alpha) \left\{ \delta \int_0^\infty [f(z-q) - f(z)]\, dG(q) + (1-\delta) \int_{\underline{q}}^\infty f(z-q)\, dG(q) \right\} \tag{33}$$
$$+ (\eta - 1) g f'(z) - \frac{\partial(\mu(z) f(z))}{\partial z} + \frac{1}{2}\sigma^2 f''(z) = 0.$$

At the activity threshold $\underline{z}$, the density $f(\underline{z}) = 0$.

The stationary distribution does not allow for an analytical solution, due to the varying innovation intensity $\mu(z)$. However, using the result in Lemma 3 that the innovation intensity converges to a constant upper bound in the limit, we can solve analytically the asymptotic distribution in the tail. When $z \to \infty$, equation (33) converges to

$$\lambda(\alpha)\left\{\delta \int_0^\infty \left[f(z-q) - f(z)\right] dG(q) + (1-\delta)\int_{\underline{q}}^\infty f(z-q)\, dG(q)\right\} \tag{34}$$
$$+\left((\eta-1)g - \bar{\mu}\right) f'(z) + \frac{1}{2}\sigma^2 f''(z) = 0.$$

The asymptotic stationary distribution in the tail has a Pareto tail: for $z > \bar{z} \to \infty$, $f(z) \propto e^{-\zeta z}$, where the tail index ζ is characterized by

$$\frac{1}{2}\sigma^2\zeta^2 + \left(\bar{\mu} - (\eta-1)g\right)\zeta + \lambda(\alpha)\left[\delta\int_0^\infty \left(e^{\zeta q}-1\right) dG(q) + (1-\delta)\int_{\underline{q}}^\infty e^{\zeta q} dG(q)\right] = 0.$$

We express the growth rate in terms of the tail index:

$$(\eta-1)g = \bar{\mu} + \frac{1}{2}\sigma^2\zeta + \frac{\lambda(\alpha)}{\zeta}\left[\delta\int_0^\infty \left(e^{\zeta q}-1\right) dG(q) + (1-\delta)\int_{\underline{q}}^\infty e^{\zeta q} dG(q)\right].$$

There can potentially exist a continuum of equilibria associated with different growth rates and stationary distributions, as in Luttmer (2007) and Benhabib et al. (2021). When the initial distribution has a bounded support under Assumption 1, the economy converges to a unique BGP, which corresponds to the one with the lowest growth rate. This occurs when $\partial g/\partial \zeta = 0$, which implies equation (17).

A.6 Intuitive Growth Accounting

In this section we conduct two growth accounting exercises, which complement the BGP analysis above based on an intuitive accounting identity. We again highlight the endogenous objects that affect the growth outcome.

A.6.1 Growth in the Tail

The first growth accounting is according to the dynamics of the firms in the tail. We carry out the following thought experiment. Consider the detrended stationary distribution $F(z)$ on the BGP. We restrict our attention to the firms in the tail, i.e., those firms with productivity $z \geq \bar{z}$, where $\bar{z} \to \infty$. Their productivity aggregates to $\bar{Z} = \int_{\bar{z}}^\infty e^z dF(z)$. We then compute the change in aggregate productivity for the firms that remain or enter the tail region over a short time interval $\Delta \to 0$. We do so for the three types of innovation

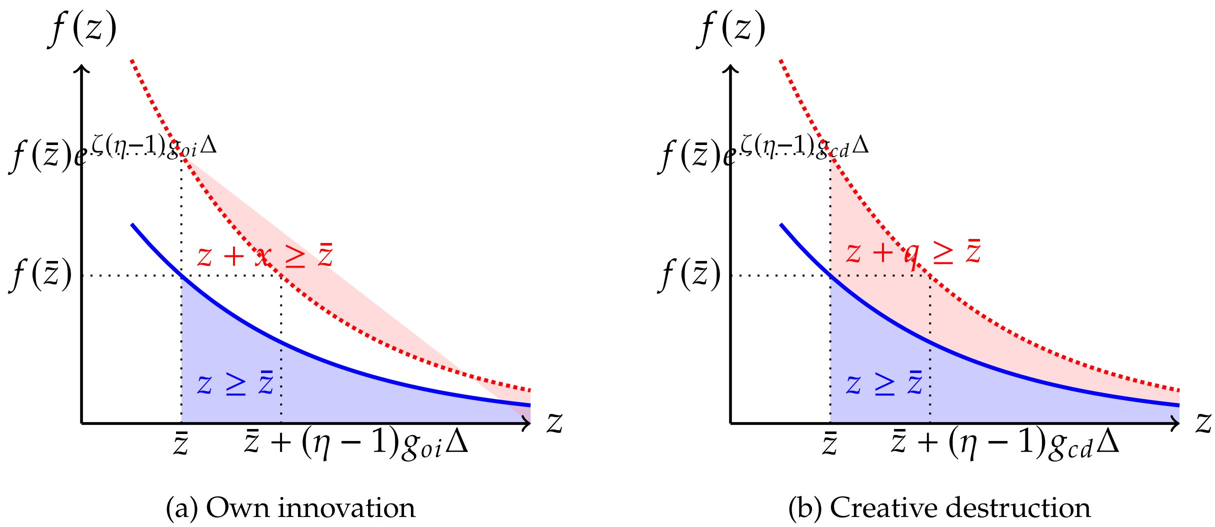


Figure 5: Events behind tail growth

Notes: Panels display how own innovation and creative destruction shift the tail of the productivity distribution.

and imitation events separately. The probability any of these events jointly happening is of higher orders of Δ, which without loss of generality can be disregarded.

First, we consider firms' own-innovation activities. One advantage of focusing on the firms in the tail is that their endogenous innovation intensity approaches the constant limit, $\mu(z) \to \bar{\mu}$. Over the time interval Δ, the outcome of own innovation can thus be represented by a productivity increment drawn from a normal distribution: $x \sim N\left(\bar{\mu}\Delta, \sigma\sqrt{\Delta}\right)$. For an innovator to remain in the tail or enter the tail, this increment must satisfy $z + x \geq \bar{z}$. These events lead to a shift in the productivity distribution depicted in panel (a) of Figure 5: accounting for the entry and exit into the tail while preserving the shape of the distribution, the distribution shifts forward by $(\eta-1)\, g_{oi}\Delta$, where g_{oi} denotes the growth due to own innovation. Naturally, the endogenous shape of the tail distribution ζ matters for how fast the tail aggregate grows. The tail aggregate scales by $e^{\zeta(\eta-1)g_{oi}\Delta}$. The growth in aggregate productivity can be computed according to

$$e^{\zeta(\eta-1)g_{oi}\Delta} = \frac{\int_{-\infty}^{\infty}\int_{\bar{z}-x}^{\infty} e^{z+x} dF(z)\, dH(x)}{\bar{Z}} = e^{\left(\zeta\bar{\mu}+\frac{1}{2}\zeta^2\sigma^2\right)\Delta}. \tag{35}$$

The second equality in the equation above follows from the transformation for the normal distribution. We thus obtain that growth from own innovation is of rate $\bar{\mu} + \frac{1}{2}\zeta\sigma^2$:

$$g_{oi} = \frac{1}{\eta - 1}\left(\bar{\mu} + \frac{1}{2}\sigma^2\zeta\right). \tag{36}$$

Next, we consider the creative-destruction events. The probability an incumbent meets an imitator who can potentially replace it is $\lambda(\alpha)\delta\Delta$. Similar to the preceding discussion for own-innovation activities, we look at those imitators that draw a quality $q \geq 0$ and $z + q \geq \bar{z}$ such that they land in the tail. The resulting change in the productivity distribution is described in panel (b) of Figure 5. Let g_{cd} denote the growth coming from these creative-destruction events:

$$e^{\zeta(\eta-1)g_{cd}\Delta} - 1 = \frac{\lambda(\alpha)\delta\Delta}{\bar{Z}} \int_0^\infty \left[\int_{\bar{z}-q}^\infty e^{z+q} dF(z) - \int_{\bar{z}}^\infty e^z dF(z) \right] dG(q). \tag{37}$$

Lastly, we consider the arrival of new varieties. For each incumbent variety, the probability of a new variety arriving is $\lambda(\alpha)(1-\delta)\Delta$. To enter the right tail, the entrant draws a quality $q \geq \underline{q}$ such that $z + q \geq \bar{z}$. The implied shift in the productivity distribution mirrors the logic in Figure 5. The resulting output growth is

$$e^{\zeta(\eta-1)g_{nv}\Delta} - 1 = \frac{\lambda(\alpha)(1-\delta)\Delta \int_{\underline{q}}^\infty \int_{\bar{z}-q}^\infty e^{z+q} dF(z)\, dG(q)}{\bar{Z}}. \tag{38}$$

As we shrink the time interval to zero, the growth rate computed in equations (37) and (38) converge to the levels in the growth decomposition earlier. We summarize these equivalent measures in the following lemma.

Lemma 4. *Consider the shift in the distribution of firms in the right tail $\bar{z} \to \infty$ over a small time interval $\Delta \to 0$. The growth rates computed according to equations (35), (37), and (38) are characterized by the three components in the growth rate equation (15) in Proposition 2.*

Own innovation. For the numerator on the right-hand side of equation (35), we perform a change of variable: $z' = z + x$. The steady state density function satisfies $f(z' - x) = e^{\zeta x} f(z')$, for $z' - x \geq \bar{z}$. Thus:

$$\begin{aligned} &\int_{-\infty}^\infty \int_{\bar{z}-x}^\infty e^{z+x} dF(z)\, dH(x) \\ &\quad = \int_{-\infty}^\infty \int_{\bar{z}}^\infty e^{z'} dF(z'-x)\, dH(x) \\ &\quad = \int_{\bar{z}}^\infty e^{z'} dF(z') \int_{-\infty}^\infty e^{\zeta x} dH(x). \end{aligned}$$

Equation (35) simplifies to

$$e^{\zeta(\eta-1)g_{oi}\Delta} = \int_{-\infty}^\infty e^{\zeta x} dH(x) = e^{\left(\zeta\bar{\mu} + \frac{1}{2}\zeta^2\sigma^2\right)\Delta},$$

from which we obtain the growth rate formula due to own innovation in equation (15).

Creative destruction. For the entry term on the right-hand side of equation (37), we perform a change of variable, $z' = z + q$. The steady state density function satisfies $f\left(z' - q\right) = e^{\zeta q} f\left(z'\right)$, for $z' - q \geq \bar{z}$. Thus:

$$\int_0^\infty \int_{\bar{z}-q}^\infty e^{z+q} dF\left(z\right) dG\left(q\right) = \bar{Z} \int_0^\infty e^{\zeta q} dG\left(q\right).$$

The displacement term is $\bar{Z} \int_0^\infty dG\left(q\right)$, because only incumbents already in the tail are removed from the tail aggregate. Hence equation (37) simplifies to

$$e^{\zeta(\eta-1)g_{cd}\Delta} - 1 = \lambda\left(\alpha\right) \delta\Delta \int_0^\infty \left(e^{\zeta q} - 1\right) dG\left(q\right).$$

Taking logs of the equation above, dividing both sides by Δ, and letting $\Delta \to 0$, we obtain the growth rate due to creative destruction in equation (15).

New varieties. Similarly, for the numerator on the right-hand side of equation (38), we perform a change of variable: $z' = z + q$. The steady state density function satisfies $f\left(z' - q\right) = e^{\zeta q} f\left(z'\right)$, for $z' - q \geq \bar{z}$. Thus:

$$\begin{aligned} &\int_{\underline{q}}^\infty \int_{\bar{z}-q}^\infty e^{z+q} dF\left(z\right) dG\left(q\right) \\ &= \int_{\underline{q}}^\infty \int_{\bar{z}}^\infty e^{z'} dF\left(z' - q\right) dG\left(q\right) \\ &= \int_{\bar{z}}^\infty e^{z'} dF\left(z'\right) \int_{\underline{q}}^\infty e^{\zeta q} dG\left(q\right). \end{aligned}$$

It simplifies to

$$e^{\zeta(\eta-1)g_{nv}\Delta} - 1 = \lambda\left(\alpha\right)\left(1-\delta\right)\Delta \int_{\underline{q}}^\infty e^{\zeta q} dG\left(q\right).$$

Taking logs of the equation above, dividing both sides by Δ, and letting $\Delta \to 0$, we obtain the growth rate due to new varieties in equation (15).

A.7 Proof of Lemma 1

Apply the Laplace transform $\int_{\underline{z}}^{\infty} e^{-\xi z} f(z)\, dz$ to the stationary KFE (33):

$$\begin{aligned}
&\left\{ \frac{1}{2}\sigma^2 \xi^2 + (\eta - 1) g\xi \right. \\
&\left. + \lambda(\alpha) \left[\delta \int_0^{\infty} \left(e^{-\xi q} - 1 \right) dG(q) + (1-\delta) \int_{\underline{q}}^{\infty} e^{-\xi q} dG(q) \right] \right\} \int_{\underline{z}}^{\infty} e^{-\xi z} f(z)\, dz \\
&- \xi \int_{\underline{z}}^{\infty} e^{-\xi z} \mu(z) f(z)\, dz = \frac{1}{2}\sigma^2 f'(\underline{z})\, e^{-\xi \underline{z}}.
\end{aligned}$$

The boundary terms use $f(\underline{z}) = 0$ and the tail decay of the stationary density. The condition $\mathbb{E}[e^z] < \infty$ allows us to evaluate this expression at $\xi = -1$:

$$\begin{aligned}
\left\{ \frac{1}{2}\sigma^2 - (\eta - 1) g + \lambda(\alpha) \left[\delta \int_0^{\infty} (e^q - 1)\, dG(q) + (1-\delta) \int_{\underline{q}}^{\infty} e^q dG(q) \right] \right\} \int_{\underline{z}}^{\infty} e^z f(z)\, dz \\
+ \int_{\underline{z}}^{\infty} e^z \mu(z) f(z)\, dz = \frac{1}{2}\sigma^2 f'(\underline{z})\, e^{\underline{z}}.
\end{aligned} \tag{39}$$

Letting $\xi = 0$ and using $\int_{\underline{z}}^{\infty} f(z)\, dz = \alpha$:

$$\lambda(\alpha)(1-\delta)\alpha = \frac{1}{2}\sigma^2 f'(\underline{z}). \tag{40}$$

Substituting equation (40) into equation (39) and dividing by $M_1 \equiv \int_{\underline{z}}^{\infty} e^z f(z)\, dz$ gives

$$\begin{aligned}
(\eta - 1) g = \frac{\int_{\underline{z}}^{\infty} e^z \mu(z) f(z)\, dz}{M_1} + \frac{1}{2}\sigma^2 + \lambda(\alpha)\delta \int_0^{\infty} (e^q - 1)\, dG(q) \\
+ \lambda(\alpha)(1-\delta) \int_{\underline{q}}^{\infty} e^q dG(q) - \lambda(\alpha)(1-\delta) \frac{\alpha e^{\underline{z}}}{M_1}.
\end{aligned}$$

Since expectations in equation (18) are taken over active firms,

$$\frac{\int_{\underline{z}}^{\infty} e^z \mu(z) f(z)\, dz}{M_1} = \frac{\mathbb{E}\left[e^z \mu(z)\right]}{\mathbb{E}[e^z]}, \qquad \frac{\alpha e^{\underline{z}}}{M_1} = \frac{e^{\underline{z}}}{\mathbb{E}[e^z]}.$$

Thus the stationary KFE implies equation (18). Proposition 2 characterizes the same BGP growth rate by the tail formula (15), so the two growth formulas coincide.

A.8 Constant Innovation Benchmark

We now collect the closed-form expressions that obtain when the innovation intensity is exogenously fixed at $\mu(z) = \bar{\mu}$. In this case the KFE (33) reduces globally to the linear differential equation (34). The stationary density is exponential, $f(z) \propto e^{-\zeta z}$, with ζ characterized by equation (17). The aggregate growth rate simplifies to the BGP expression in (15), where the diffusion component becomes $\lambda(\alpha) D(\zeta)$ evaluated at the constant upper bound. These formulas provide the basis for the comparative statics highlighted in Section B and for the reference made in the main text.

A.9 Derivation of Example 1

Example 1 (Exponential quality distribution)**.** Suppose the quality draw for knowledge diffusion follows an exponential distribution:

$$G(q) = 1 - e^{-\phi(q-\underline{q})}, \ \forall q \geq \underline{q}.$$

The diffusion function in equation (15) becomes

$$D(\zeta) = \frac{1}{\zeta}\left[\delta e^{\phi\underline{q}} \frac{\zeta}{\phi-\zeta} + (1-\delta) e^{\zeta\underline{q}} \frac{\phi}{\phi-\zeta}\right]. \tag{41}$$

Equation (17) that characterizes the Pareto tail index becomes

$$\lambda(\alpha)\left((1-\delta) e^{\zeta\underline{q}} \frac{\phi\left(\phi - 2\zeta - \underline{q}\zeta(\phi-\zeta)\right)}{\zeta^2(\phi-\zeta)^2} - \delta e^{\phi\underline{q}} \frac{1}{(\phi-\zeta)^2}\right) = \frac{1}{2}\sigma^2. \tag{42}$$

Under the conditions in Lemma 3, when the quality draw follows an exponential distribution, we have $\int_{\underline{q}}^{\infty} e^q dG(q) = e^{\phi\underline{q}} \frac{\phi}{\phi-1}$, $\int_0^{\infty} (e^q - 1) dG(q) = e^{\phi\underline{q}} \frac{1}{\phi-1}$, $\int_{\underline{q}}^{\infty} e^{\zeta q} dG(q) = e^{\phi\underline{q}} \frac{\zeta}{\phi-\zeta}$, and $\int_{\underline{q}}^{\infty} (e^{\zeta q} - 1) dG(q) = e^{\phi\underline{q}} \frac{\phi}{\phi-\zeta}$. Given these expressions, we obtain that

$$D(1) = \delta e^{\phi\underline{q}} \frac{1}{\phi-1} + (1-\delta) e^{\phi\underline{q}} \frac{\phi}{\phi-1}.$$

Further, $\int_0^{\infty} e^{\zeta q} q dG(q) = e^{\phi\underline{q}} \frac{\phi}{(\phi-\zeta)^2}$ and $\int_{\underline{q}}^{\infty} e^{\zeta q} q dG(q) = \frac{\phi}{\phi-\zeta}\left(\underline{q} + \frac{1}{\phi-\zeta}\right) e^{\zeta\underline{q}}$. Substituting these expressions into equation (17), we obtain equation (42).

All productivity heterogeneity is driven by imitation.

Lemma 5 (Deterministic Innovation)**.** *In Example 1, suppose that the innovation outcome is deterministic, i.e.,* $\sigma = 0$*. Specifically, the only source of randomness is imitation:* $\underline{q} = 0$ *and* $\phi > 1$*. Along the BGP, the economy grows at rate*

$$g = \frac{1}{\eta - 1}\left(\bar{\mu} + \frac{1}{\phi}\lambda\left(\alpha\right)\left(\sqrt{1-\delta}+1\right)^2\right), \tag{43}$$

and the endogenous stationary distribution has a Pareto tail index

$$\zeta = \phi\frac{\sqrt{1-\delta}}{\sqrt{1-\delta}+1}. \tag{44}$$

Proof. When $\sigma = 0$, equation (42) for the tail index simplifies to

$$\frac{1-\delta}{\zeta^2} - \frac{1}{\left(\phi - \zeta\right)^2} = 0.$$

Solving for the equation above, we obtain two roots: $\zeta_1 = \phi\sqrt{1-\delta}/\left(\sqrt{1-\delta}+1\right)$ and $\zeta_2 = \phi\sqrt{1-\delta}/\left(\sqrt{1-\delta}-1\right)$. Taking the positive root, we obtain the tail index in equation (44). Letting $\sigma = 0$ and substituting the tail index into (41), we obtain equation (43). □

A.10 Proof of Lemma 2

When $\phi = \infty$ and $\underline{q} = 0$, equation (42) for the tail index simplifies to

$$\lambda\left(\alpha\right)\frac{1-\delta}{\zeta^2} = \frac{1}{2}\sigma^2.$$

Solving for the equation above and taking the positive root, we obtain the tail index in equation (19). Letting $\phi = \infty$ and substituting the tail index in (19) into (41), we obtain equation (20).

A.11 Social Value Function

Let $V^s\left(z,t\right)$ denote the social value function associated with an innovator with productivity z at time t. Let $W^s\left(t\right)$ denote the social value function associated with an imitator at time t. The social value functions satisfy the following HJB equations:

$$
\begin{aligned}
r(t) V^s(z,t) =& \Pi(t)\left(1-c\left(\mu\right)\right)e^z + \lambda\left(\alpha(t)\right) S(z,t) \\
& + \mu V^s_z(z,t) + \frac{1}{2}\sigma^2 V^s_{zz}(z,t) + V^s_t(z,t), \\
r(t) W^s(t) =& \phi\left(\alpha(t)\right) \int_{\underline{z}(t)} S(z,t)\, dF(z,t) + W^s_t(t).
\end{aligned} \tag{45}
$$

On the BGP, equation (45) simplifies to

$$
\rho V^s(z) = \pi\left(1-c\left(\mu\right)\right)e^z + \lambda(\alpha) S(z) + \left(\mu - \left(\eta-1\right)g\right) V^s_z(z) + \frac{1}{2}\sigma^2 V^s_{zz}(z).
$$

Taking the limit $z \longrightarrow \infty$, we obtain $\lim_{z\to\infty} \frac{V^s(z)}{e^z} = \bar{V}^s$, where

$$
\bar{V}^s = \frac{1 - c\left(\bar{\mu}\right)}{\rho + \left(\eta - 1\right)g - \bar{\mu} - \frac{1}{2}\sigma^2 - \lambda(\alpha) D(1)}. \tag{46}
$$

When q follows an exponential distribution, substituting $\int e^q dq = \frac{\phi}{\phi-1}$ into the expression above, we obtain the marginal social value function in equation (46).

B Social Planner's Problem

The problem in Section 5.3 is defined as follows. Given the initial distribution $F(z,0)$, the planner chooses the path of policy $\left\{\beta(t)\right\}_{t\geq 0}$ and equilibrium outcome $\{S^e(z,t)$, $W(t)$, $\underline{z}(t)$, $\alpha(t)$, $\{V(z,t), \mu(z,t), F(z,t)\}_{z>\underline{z}(t)}\Big\}_{t\geq 0}$ to maximize the welfare $\int_0^\infty e^{-\rho t}\log\left(C(t)\right) dt$ subject to the equilibrium conditions (7) to (13), final good market clearing condition $C(t) = Y(t)$, and the Euler equation. When we normalize the total expenditure $P(t)C(t)$ to 1, the aggregate profit is given by $\Pi(t) = 1/\left(\eta Z(t)\right)$, and the Euler equation gives $r(t) = \rho$.

Let $u(t)$ denote the fraction of imitators at time t, where

$$
u(t) = 1 - \alpha(t) \tag{47}
$$

The KFE for innovators (13) and the definition of $\alpha(t)$ (6) imply the law of motion of the fraction of imitators:

$$
u_t(t) = -m\left(\alpha(t), u(t)\right) + \frac{1}{2}\sigma^2 f_z(z,t). \tag{48}
$$

We include (47) and (48) as a constraint of the planner instead of (6), because the interpretation of the costate variables will turn out to be more intuitive.

Now, we construct the current value Hamiltonian of this problem. To do this, we first define the sets of state variables $\mathbf{x}(t)$, control variables $\mathbf{y}(t)$, and costate variables $\boldsymbol{\psi}(t)$ as follows.

$$\begin{aligned}
\mathbf{x}(t) &= \left[\left\{ f(z,t), V(z,t) \right\}_{z>\underline{z}(t)} \right] \\
\mathbf{y}(t) &= \begin{bmatrix} \beta(t), \underline{z}(t), \alpha(t), u(t), W(t), \\ \left\{ \mu(z,t) \right\}_{z>\underline{z}(t)} \end{bmatrix} \\
\boldsymbol{\psi}(t) &= \begin{bmatrix} \psi^{Z}(t), \psi^{\alpha}(t), \psi^{bc}(t), \psi^{u}(t), \psi^{w}(t), \psi^{vm}(t), \psi^{vp}(t), \\ \left\{ \psi^{f}(z,t), \psi^{v}(z,t), \psi^{\mu}(z,t) \right\}_{z>\underline{z}(t)} \end{bmatrix}
\end{aligned}$$

Then, the current value Hamiltonian is given by

$$
\begin{aligned}
&\mathcal{H}\left(\mathbf{x}(t), \mathbf{y}(t), \boldsymbol{\psi}(t)\right) = \\
&\quad \log\left(Z\left(t\right)\right) \\
&\quad + \psi^{Z}\left(t\right)\left\{\int_{\underline{z}(t)}^{\infty}\left(1 - c\left(\mu\left(z, t\right)\right)\right)\frac{e^{z}}{Z(t)} f\left(z, t\right) dz - 1\right\} \\
&\quad + \psi^{\alpha}\left(t\right)\left\{\alpha\left(t\right) + u\left(t\right) - 1\right\} \\
&\quad + \int_{\underline{z}(t)}^{\infty} \psi^{f}\left(z, t\right)\left[\begin{array}{c} \lambda\left(\alpha\left(t\right)\right)\left\{\begin{array}{c} \delta\int_{0}^{\infty}\left[f\left(z - q, t\right) - f\left(z, t\right)\right] dG\left(q\right) \\ +\left(1 - \delta\right)\int_{\underline{q}}^{\infty} f\left(z - q, t\right) dG\left(q\right) \end{array}\right\} \\ -\frac{\partial}{\partial z}\left(\mu\left(z, t\right) f\left(z, t\right)\right) + \frac{1}{2}\sigma^{2} f_{zz}\left(z, t\right) \end{array}\right] dz \\
&\quad + \psi^{bc}\left(t\right) f\left(\underline{z}\left(t\right), t\right) \\
&\quad + \psi^{u}\left(t\right)\left[-m\left(\alpha\left(t\right), u\left(t\right)\right) + \frac{1}{2}\sigma^{2} f_{z}\left(z, t\right)\right] dt \\
&\quad + \int_{\underline{z}(t)}^{\infty} \psi^{v}\left(z, t\right)\left[\begin{array}{c} \rho V\left(z, t\right) - \frac{1}{\eta}\left(1 - c\left(\mu\right)\right)\frac{e^{z}}{Z(t)} - \mu V_{z}\left(z, t\right) - \frac{1}{2}\sigma^{2} V_{zz}\left(z, t\right) \\ -\lambda\left(\alpha\left(t\right)\right)\beta\left\{\begin{array}{c} \delta\int_{q \geq 0}(V(z + q, t) - W(t)) dG(q) \\ +(1 - \delta)\int_{z + q \geq \underline{z}(t)}(V(z + q, t) - W(t)) dG(q) \end{array}\right\} \\ +\lambda\left(\alpha\left(t\right)\right)\tilde{\delta}\left(V\left(z, t\right) - W\left(t\right)\right) \end{array}\right] dz \\
&\quad + \int_{\underline{z}(t)}^{\infty} \psi^{w}\left(t\right)\left[\rho W\left(t\right) - \iota\left(\alpha\left(t\right)\right)\left(1 - \beta\right)\mathbb{E}\left[S^{e}\left(z, t\right)\right]\right] dz \\
&\quad + \psi^{vm}\left(t\right)\left\{V\left(\underline{z}\left(t\right), t\right) - W\left(t\right)\right\} \\
&\quad + \psi^{vp}\left(t\right) V_{z}\left(\underline{z}\left(t\right), t\right) \\
&\quad + \int_{\underline{z}(t)}^{\infty} \psi^{\mu}\left(z, t\right)\left\{\frac{1}{\eta} c'\left(\mu\left(z, t\right)\right)\frac{e^{z}}{Z(t)} - V_{z}\left(z, t\right)\right\} dz
\end{aligned}
$$

We rewrite the fourth term on the right-hand side of the Hamiltonian using integration by parts:

$$\int_{\underline{z}(t)}^{\infty} \psi^f(z,t) \left[\begin{array}{c} \lambda(\alpha(t)) \left\{ \begin{array}{c} \delta \int_0^\infty \left[f(z-q,t) - f(z,t) \right] dG(q) \\ +(1-\delta) \int_{\underline{q}}^\infty f(z-q,t)\, dG(q) \end{array} \right\} \\ -\frac{\partial}{\partial z}\left(\mu(z,t) f(z,t) \right) + \frac{1}{2}\sigma^2 f_{zz}(z,t) \end{array} \right] dz$$

$$= \int_{\underline{z}(t)}^{\infty} \psi^f(z,t)\, \lambda(\alpha(t)) \left\{ \begin{array}{c} \delta \int_0^\infty \left[f(z-q,t) - f(z,t) \right] dG(q) \\ +(1-\delta) \int_{\underline{q}}^\infty f(z-q,t)\, dG(q) \end{array} \right\} dz$$

$$+ \int_{\underline{z}(t)}^{\infty} f(z,t) \left\{ \mu(z,t)\, \psi_z^f(z,t) + \frac{1}{2}\sigma^2 \psi_{zz}^f(z,t) \right\} dz$$

$$- \psi^f(\infty,t)\, \mu(\infty,t)\, f(\infty,t)$$

$$+ \psi^f\left(\underline{z}(t),t\right) \mu\left(\underline{z}(t),t\right) f\left(\underline{z}(t),t\right)$$

$$+ \frac{1}{2}\sigma^2 \left\{ -\psi_z^f(\infty,t)\, f(\infty,t) + \psi_z^f\left(\underline{z}(t),t\right) f\left(\underline{z}(t),t\right) \right.$$

$$\left. +\psi^f(\infty,t)\, f_z(\infty,t) - \psi^f\left(\underline{z}(t),t\right) f_z\left(\underline{z}(t),t\right) \right\}$$

As a result of this procedure, it becomes straightforward to take the derivative with respect to $f(z,t)$. We also rewrite the sixth and last terms on the right-hand side of the Hamiltonian using integration by parts.

$$\int_{\underline{z}(t)}^{\infty} \psi^v(z,t) \left[\begin{array}{c} \rho V(z,t) - \frac{1}{\eta}\left(1 - c(\mu)\right) \frac{e^z}{Z(t)} - \mu V_z(z,t) - \frac{1}{2}\sigma^2 V_{zz}(z,t) \\ -\lambda(\alpha(t))\, \beta \left\{ \begin{array}{c} \delta \int_{q \geq 0} (V(z+q,t) - W(t)) dG(q) \\ +(1-\delta) \int_{z+q \geq \underline{z}(t)} (V(z+q,t) - W(t)) dG(q) \end{array} \right\} \\ +\lambda(\alpha(t))\, \tilde{\delta} \left(V(z,t) - W(t) \right) \end{array} \right] dz$$

$$= \int_{\underline{z}(t)}^{\infty} \psi^v(z,t) \left[\begin{array}{c} \rho V(z,t) - \frac{1}{\eta}\left(1 - c(\mu)\right) \frac{e^z}{Z(t)} \\ -\lambda(\alpha(t))\, \beta \left\{ \begin{array}{c} \delta \int_{q \geq 0} (V(z+q,t) - W(t)) dG(q) \\ +(1-\delta) \int_{z+q \geq \underline{z}(t)} (V(z+q,t) - W(t)) dG(q) \end{array} \right\} \\ +\lambda(\alpha(t))\, \tilde{\delta} \left(V(z,t) - W(t) \right) \end{array} \right] dz$$

$$+ \int_{\underline{z}(t)}^{\infty} V(z,t) \left\{ \frac{\partial}{\partial z}\left(\mu(z,t)\, \psi^v(z,t) \right) - \frac{1}{2}\sigma^2 \psi_{zz}^v(z,t) \right\} dz$$

$$- \mu(\infty,t)\, \psi^v(\infty,t)\, V(\infty,t)$$

$$+ \mu\left(\underline{z}(t),t\right) \psi^v\left(\underline{z}(t),t\right) V\left(\underline{z}(t),t\right)$$

$$- \frac{1}{2}\sigma^2 \left[-\psi_z^v(\infty,t)\, V(\infty,t) + \psi_z^v\left(\underline{z}(t),t\right) V\left(\underline{z}(t),t\right) \right.$$

$$\left. +\psi^v(\infty,t)\, V_z(\infty,t) - \psi^v\left(\underline{z}(t),t\right) V_z\left(\underline{z}(t),t\right) \right]$$

and

$$\int_{\underline{z}(t)}^{\infty} \psi^{\mu}(z,t) \left\{ \frac{1}{\eta} c'\left(\mu(z,t)\right) \frac{e^{z}}{Z(t)} - V_{z}(z,t) \right\} dz$$
$$= \int_{\underline{z}(t)}^{\infty} \left[\psi^{\mu}(z,t) \frac{1}{\eta} c'\left(\mu(z,t)\right) \frac{e^{z}}{Z(t)} + \psi_{z}^{\mu}(z,t) V(z,t) \right] dz$$
$$+ \psi^{\mu}(\infty,t) V(\infty,t) - \psi^{\mu}\left(\underline{z}(t),t\right) V\left(\underline{z}(t),t\right)$$

As a result of this procedure, it becomes straightforward to take the derivative with respect to $f(z,t)$.

Given the Hamiltonian, the social planner solution satisfies the following optimality conditions

$$\frac{\partial}{\partial \mathbf{y}(t)} \mathcal{H}\left(\mathbf{x}(t), \mathbf{y}(t), \boldsymbol{\psi}(t)\right) = 0$$
$$\frac{\partial}{\partial \mathbf{x}(t)} \mathcal{H}\left(\mathbf{x}(t), \mathbf{y}(t), \boldsymbol{\psi}(t)\right) = \rho \mathbf{x}(t) - \frac{d}{dt}\mathbf{x}(t)$$

and all the constraints. In the following, we focus on the stationary version of the social planner's problem.

Let us rename ψ^{f} as $\tilde{V}$ because the costate variable of the KFE for innovators corresponds to their social values. Similarly, let us rename ψ^{u} as $\tilde{W}$ because the costate variable of the law of motion for the fraction of imitators corresponds to the social value of an imitator. Given the social values $\tilde{V}$ and $\tilde{W}$, let us define the social match surplus:

$$\tilde{S}(z) = \delta \int_{0}^{\infty} \left[\tilde{V}(z+q) - \tilde{V}(z) \right] dG(q) + (1-\delta) \int_{\max\left\{\underline{q}, \underline{z}-z\right\}}^{\infty} \left[\tilde{V}(z+q) - \tilde{W} \right] dG(q). \quad (49)$$

Rearranging the optimality conditions, we obtain the HJB equations for the social value functions of innovators and imitators:

$$\rho \tilde{V}(z) = \frac{1}{\eta-1} Z^{\frac{2-\eta}{\eta-1}} L\left(1 - c(\mu)\right) e^{z} + \left(\mu(z) - (\eta-1) g\right) \tilde{V}_{z}(z) + \frac{1}{2}\sigma^{2} \tilde{V}_{zz}(z)$$
$$+ \lambda(\alpha) \left\{ \tilde{S}(z) - (1-\omega)\mathbb{E}\left[\tilde{S}(z)\right] + \frac{\psi^{w}}{1-\alpha}\left((1-\beta) S(z) - (1-\omega)\mathbb{E}\left[S(z)\right]\right) \right.$$
$$\left. + (1-\omega)\tilde{\delta} \int_{\underline{z}}^{\infty} \frac{\psi^{v}(\tilde{z})}{\alpha} \left(V(\tilde{z}) - W\right) d\tilde{z} \right\},$$
$$\rho \tilde{W} = \gamma(\alpha) \left\{ (1-\omega)\mathbb{E}\left[\tilde{S}(z)\right] + \frac{\psi^{w}}{1-\alpha}(\beta-\omega)\mathbb{E}\left[S(z)\right] - (1-\omega)\tilde{\delta} \int_{\underline{z}}^{\infty} \frac{\psi^{v}(\tilde{z})}{\alpha}\left(V(\tilde{z}) - W\right) d\tilde{z} \right\}.$$

with value matching and smooth pasting conditions:

$$\tilde{V}\left(\underline{z}\right) = \tilde{W},$$
$$\tilde{V}_z\left(\underline{z}\right) = 0.$$

These HJB equations, value matching conditions, and smooth pasting conditions are analogous to the private ones but include the external effects on matching. These matching externalities include the value of the match for both parties and the crowding out externalities due to matching.

In addition to the value functions and all the constraints for the planner, the remaining four conditions obtained from the optimality conditions are required to pin down the multipliers $\psi^v(z)$, $\psi^\mu(z)$, ψ^w, and the optimal bargaining weight β:

$$\psi^\mu(z) = \frac{\tilde{V}_z(z) - \frac{\eta}{\eta-1}V_z(z)}{\Pi c''\left(\mu(z)\right)e^z} f(z).$$

$$\begin{aligned}
0 = {} & \lambda(\alpha)\left\{\beta\left[\delta\int_0^{z-\underline{z}}\psi^v\left(z-q\right)dG\left(q\right) + (1-\delta)\int_{\underline{q}}^{z-\underline{z}}\psi^v\left(z-q\right)dG\left(q\right)\right] - \tilde{\delta}\psi^v(z)\right\} \\
& + \lambda(\alpha)\left(1-\beta\right)\frac{\psi^w}{1-\alpha}\left[\delta\int_0^{z-\underline{z}} f\left(z-q\right)dG\left(q\right) + (1-\delta)\int_{\underline{q}}^{z-\underline{z}} f\left(z-q\right)dG\left(q\right)\right] \\
& + \left(\eta-1\right)g\psi_z^v(z) - \frac{\partial\left(\mu(z)\psi^v(z)\right)}{\partial z} + \frac{1}{2}\sigma^2\psi_{zz}^v(z) - \psi_z^\mu(z).
\end{aligned}$$

$$\int_{\underline{z}}^{\infty}\psi^v(z)\,dz + \psi^w = 0.$$

$$\int_{\underline{z}}^{\infty} S^e(z)\frac{\psi^v(z)}{\alpha}dz = \frac{\psi^w}{1-\alpha}\mathbb{E}\left[S^e(z)\right]. \tag{50}$$

In summary, the social planner allocation is characterized by the equilibrium conditions (7) to (13) and the optimality conditions (49) to (50)

C Scalable-Volatility Innovation Process

For the scalable-volatility specification, most of the equations characterizing the equilibrium are unchanged. We modify a few equations where the volatility specification matters.

The HJB equation (8) for the innovator's value function becomes

$$r(t)V(z,t) = \max_{\mu}\Big\{\Pi(t)\left(1-c(\mu)\right)e^{z} + \beta\lambda(\alpha(t))S(z,t) + \mu V_z(z,t) + \frac{1}{2}\sigma(\mu)^2 V_{zz}(z,t) + V_t(z,t)\Big\}$$

The optimality condition (10) for innovation now takes into account the effects on not only the upward drift but also the volatility:

$$\Pi(t)c'\left(\mu(z,t)\right)e^{z} = V_z(z,t) + \frac{1}{2}\sigma^2 V_{zz}(z,t).$$

The KFE (13) for the distribution of firm productivity is modified to

$$f_t(z,t) = \lambda(\alpha(t))\left\{\delta\int_0^{\infty}\left[f(z-q,t)-f(z,t)\right]dG(q) + (1-\delta)\int_{\underline{q}}^{\infty} f(z-q,t)\,dG(q)\right\} - \frac{\partial\left(\mu(z,t)f(z,t)\right)}{\partial z} + \frac{1}{2}\frac{\partial^2\left(\sigma^2\mu(z,t)f(z,t)\right)}{\partial z^2}. \tag{51}$$

Along the BGP, the growth rate due to own innovation (36) derived from the KFE (51) is modified to

$$g_{oi} = \frac{1}{\eta-1}\left(\bar{\mu} + \frac{1}{2}\sigma^2\bar{\mu}\zeta\right)$$

where the upper bound of innovation $\bar{\mu}$ is characterized by

$$c'(\bar{\mu}) = \frac{\left(1+\frac{1}{2}\sigma^2\right)\left(1-c(\bar{\mu})\right)}{\rho + (\eta-1)g - \bar{\mu} - \frac{1}{2}\sigma^2\bar{\mu} - \beta D(1)}.$$

Finally, the endogenous stationary distribution, $\forall z > \underline{z}$, has a Pareto tail with tail index

$$\zeta = \frac{\sqrt{2(1-\delta)\lambda(\alpha)}}{\sigma\sqrt{\bar{\mu}}}.$$