Inventor Mobility, Knowledge Diffusion, and Growth[1]

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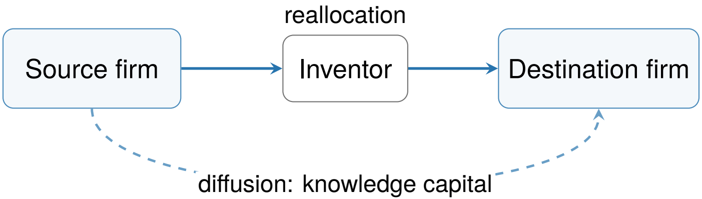
[1] The views expressed here are not those of the Bank of Japan.

When Does Inventor Mobility Raise Welfare?

"Mobility of personnel among firms provides a way of spreading information." Arrow (1962)



Growth benefits

- Reallocation: destination gains an inventor
- Diffusion: destination absorbs source knowledge

Welfare tradeoff

- Search is costly
- Search and innovation externalities distort mobility

Empirical Facts Motivate the Model

Data

- German employer–employee matched data linked to patent records

Empirical Results

- Patent-listed inventors often move to lower-mean-wage establishments
- Inventors experience larger own-wage gains when they move

Model

- An endogenous growth model with on-the-job search and knowledge diffusion
- Knowledge diffusion reshapes job ladders and inventor reallocation

Counterfactual Exercises and Main Results

Growth Effects

- Non-compete ban increases job flows, raising growth through (i) better inventor allocation and (ii) knowledge diffusion
- Knowledge diffusion drives most of the growth effects

Welfare Tradeoff

- Non-compete ban: consumption-equivalent welfare falls by 0.38%
- A near-50–50 bargaining split attains a welfare level very close to the planner's welfare

Many Inventor Flows Go to Lower-Mean-Wage Establishments

Patent-listed inventor job-to-job flows
Rank by establishment mean wage

	Destination rank bin									
Origin	1	2	3	4	5	6	7	8	9	10
1	18	16	13	12	10	9	7	6	5	4
2	11	16	14	12	10	9	8	8	7	6
3	7	13	14	12	10	9	9	9	8	8
4	6	10	11	12	11	11	11	10	10	8
5	5	8	9	11	11	11	10	12	11	12
6	4	7	9	9	10	12	13	12	12	11
7	4	6	7	8	9	11	14	13	14	15
8	3	5	6	7	10	11	14	13	14	17
9	2	5	5	6	9	9	14	14	16	21
10	2	3	4	4	6	7	10	12	19	32

- Bins rank establishments by mean wage, from 1 to 10
- 36% of patent-listed inventor job-to-job flows go to lower establishment mean-wage-rank bins
- Frequent downward flows suggest forces beyond simple wage-ladder climbing

Robustness

Inventor-Proxy Movers Move to Lower-Mean-Wage Establishments Yet Gain Wages

	Full job-to-job sample	
Dependent variable	Lower-mean-wage dest.	Own-wage growth
	(1)	(2)
Inventor-proxy mover	0.0080***	0.0581***
	(0.0021)	(0.0033)
Demographic controls	Yes	Yes
Prior wage controls	Yes	Yes
Origin and destination FE	Yes	Yes
Year FE	Yes	Yes
Observations	651,554	651,554

- Relative to other job-to-job movers, inventor-proxy movers are more likely to move to lower-mean-wage establishments and receive larger own-wage gains

▸ Wage-increase robustness ▸ SIAB/INV-BIO summary

These Facts Motivate a Knowledge-Carrying Search Model

Wage-ladder view

In a simple wage-ladder on-the-job search model, job-to-job moves should be upward on the wage ladder.

What we see

Inventors often move down the establishment mean-wage ladder, while inventor-proxy movers still gain wages.

Knowledge channel

The hiring firm can value not only the inventor, but also the knowledge carried by the inventor.

Model Setup: Search, Reallocation, and Knowledge Diffusion

Firm State

- A firm employs inventors n
- A firm owns product lines (knowledge capital) k
- Both n and k are continuous masses
- Static payoff is πk
- The payoff-relevant state is $x \equiv n/k$

Vacancy Posting and Poaching Rule

- Vacancies meet employed inventors through random on-the-job search
- Firms choose vacancies $v(x)$ to maximize the joint value of the firm and incumbent inventors
- A move occurs only when joint poaching surplus is positive

▸ Formal setup

Knowledge Dynamics

- Continuous time
- Internal R&D expands k at rate $\mu(x)$
- k is lost to creative destruction at rate $\delta(t)$
- k is subject to proportional Brownian shocks
- $x^{-1} = k/n$: inventor productivity

State Changes After a Move

When an inventor moves from source x' to destination x, the destination firm:

1. increases its inventor stock: $\Delta n = 1$
2. replicates source knowledge: $\Delta k = \alpha(x, x')$

Knowledge Diffusion Makes Downward Moves Profitable

- $S(x)$ is the joint value per product line of a firm and its incumbent inventors, where $x = n/k$ is incumbent inventor mass per product line
- $M(x, x')$ is the match surplus from moving an inventor from source firm (x') to destination firm (x)

$$M(x,x') = \max\Big\{ \underbrace{S_x(x)}_{\text{destination inventor value}} + \underbrace{\alpha(x,x')\{S(x) - xS_x(x)\}}_{\text{knowledge diffusion value}} - \underbrace{S_x(x')}_{\text{source inventor value}}, 0 \Big\}$$

$S_x(x)$ is the marginal joint value of an inventor; $S(x) - xS_x(x)$ is the marginal joint value of knowledge capital.

No diffusion

- Match surplus compares only inventor values
- Poaching is profitable only when $S_x(x) > S_x(x')$ (equiv. $x^{-1} > x'^{-1}$)

With diffusion

- Destination values source knowledge in addition to the inventor
- Diffusion can make $M(x, x') > 0$ even when $S_x(x) < S_x(x')$

▸ Firm-Level HJB

Weaker Non-Competes Raise Poaching Incentives

Three-Party Bargaining

The poaching firm receives $(1-\beta)M$; the source-side coalition (the source firm and the inventor being poached) receives βM. The split within the coalition is left unspecified.

$$\underbrace{c'(v(x))}_{\text{marginal vacancy cost}} = \underbrace{(1-\beta)}_{\text{poacher share}} \cdot \underbrace{q(\theta)}_{\text{meeting rate}} \cdot \underbrace{\int M(x,x')\, dF_x(x')}_{\text{expected match surplus}}$$

$\theta = V/N$ (V: vacancies, N: inventors); $q(\theta)$: meeting rate per vacancy; F_x: inventor-weighted.

Non-Compete Mechanism

- Stronger non-compete enforceability raises the source coalition's surplus share β (Shi, 2023)
- Weaker non-competes raise the poacher share $1-\beta$: vacancy posting $v(x)\uparrow$

Mobility Shapes Growth Through Allocation and Diffusion

$$g = \underbrace{\log(\lambda)}_{\text{innovation step}} \times \underbrace{(\delta_{RD} + \delta_{KD})}_{\text{creative destruction rate}}$$

Allocation margin

$$\delta_{RD} = \int \mu(x) dF(x)$$

With concave internal R&D, dispersion in inventor allocation lowers growth.

Diffusion margin

$$\delta_{KD} = N\theta q(\theta) \iint \mathbb{1}_P(x, x')\alpha(x, x')\, dF_x(x')dF_v(x)$$

Successful poaching moves source knowledge into destination firms.

F is product-line-weighted; F_x and F_v are inventor- and vacancy-weighted; $\mathbb{1}_P = 1$ for profitable poaching.

Sources of Private-Social Wedges

Private-social wedge	Source	Effect on private incentives
Innovation value	Appropriability	Too little private innovation value
Innovation value	Business stealing	Too much private innovation value
Vacancy posting	Hold-up from surplus sharing	Too few vacancies
Vacancy posting	Search congestion	Too many vacancies

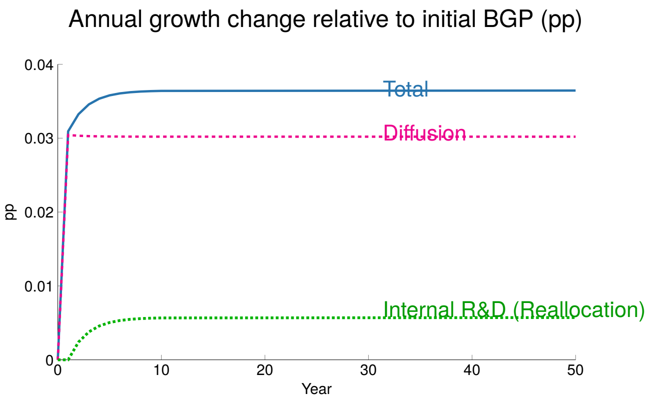
▸ Innovation value ▸ Vacancy margin ▸ Hosios

Calibration Disciplines Diffusion, R&D, and Baseline Mobility

Block	Parameters	Source or target
External or normalized	Discount $\rho = 0.004$; vacancy cost $\bar{c} = 1$, $\phi = 3.45$; matching elasticity $\eta = 0.50$; R&D elasticity $\epsilon = 0.50$; profit $\pi = 0.23$; inventor mass $N = 1.95$	Normalizations, literature values, markup, inventor mass
Productivity-growth estimates	Diffusion scale $\overline{\alpha} = 0.526$; R&D scale $\overline{\mu} = 0.0212$; shock s.d. $\sigma = 0.203$	Productivity-growth regression coefficients and RMSE
Baseline BGP targets	Matching efficiency $\overline{q} = 0.0198$; innovation step $\lambda = 1.053$	2001 inventor mobility and 2% annual aggregate growth
Policy benchmark	$\beta = 0.5$	Symmetric Nash benchmark

▸ Identification ▸ Regression ▸ Conversions

A Non-Compete Ban Raises Growth Mainly Through Diffusion



Non-Compete Ban

$\beta : 0.5 \longrightarrow 0$ at year 0; long-run annual growth rises by 0.036 pp.

Decomposition

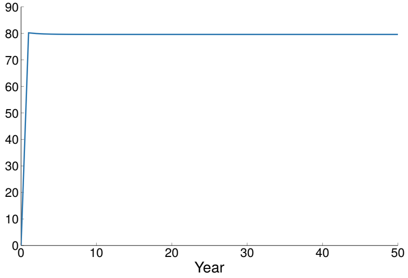
- Diffusion: 0.030 pp
- Internal R&D (Reallocation): 0.006 pp

Why Small Reallocation Effect?

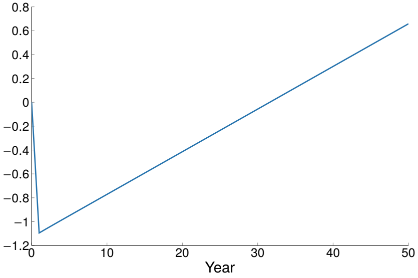
Diffusion makes job flows not one-way toward higher S_x firms.

A Non-Compete Ban Raises Growth but Lowers Welfare Through Vacancy Costs

(a) Vacancy-cost-to-output ratio relative to initial BGP (%)



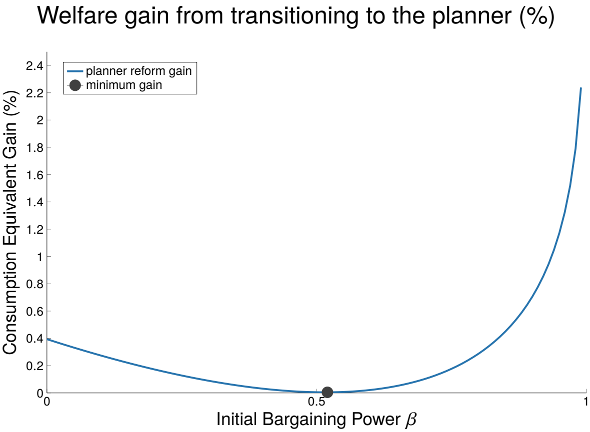
(b) log C relative to initial BGP (%)



- Long-run growth rises after the ban
- Vacancy costs jump by about 80%, lowering consumption on impact
- The initial consumption loss outweighs later growth gains: CE welfare falls by 0.38%

▸ Welfare formula ▸ Vacancy response

A Fixed Split Near 50–50 Nearly Implements the Planner

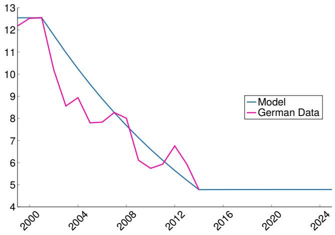


- For each bargaining power β, compute the maximum attainable CE welfare gain by transitioning from its SS to the planner allocation
- Best fixed bargaining power: $\beta = 0.52$, with a planner-reform gain of 0.0046%

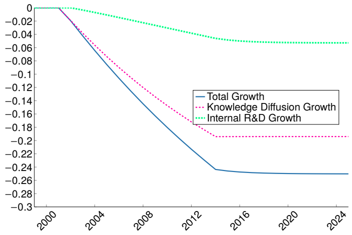
Hosios

Mobility Decline Lowers Growth Mainly by Weakening Diffusion

(a) Observed mobility path



(b) Growth loss decomposition (pp)



Mobility shock

Choose time-varying matching efficiency $\overline{q}_t$ to match the 2001, 2007, and 2014 inventor job-flow rates; then hold it fixed.

Aggregate Effects

- Annual growth: 0.25 pp $\downarrow$
- CE welfare: 3.49% loss

Decomposition

- Diffusion: 0.19 pp $\downarrow$
- Internal R&D: 0.05 pp $\downarrow$

▸ 1999–2007 ▸ 2008–2014 ▸ U.S. ▸ Welfare formula

Conclusion

Evidence

- Inventor mobility carries knowledge across establishments
- Downward moves and wage gains point beyond a simple wage-ladder model

Model

- Inventor mobility raises growth through diffusion and reallocation
- Inventor job flows respond endogenously to poaching incentives

Quantification

- Non-compete ban: growth rises, CE welfare falls by 0.38%
- Mobility decline: annual growth falls by 0.25 pp; CE welfare falls by 3.49%
- A near-50–50 bargaining split attains a welfare level very close to the planner's welfare

Appendix Roadmap

Empirical	Calibration	Policy	Theory
Data construction, inventor-proxy occupations, robustness checks, and mobility evidence.	Functional forms, parameter mapping, and annual-to-monthly conversions.	Welfare calculation and second-best bargaining details.	Competitive equilibrium, firm growth, and planner conditions.

INV-BIO Links Patent Inventors to German Employer Records

- **German linked inventor biography data (INV-BIO)**
  - Patent-register data from EPO, supplemented with DPMAregister
  - Linked to German employer-employee records
- Labor-market information from 1980–2014
- 152,350 inventors listed on EPO patent filings between 1999–2011
- Establishment information includes employment, mean daily wage, inventor productivity, and inventor job flows

Summary Statistics: INV-BIO

Establishment-level variables	Mean	S.D.	N, thousand
N of inventors (n_{et})	5.2	29.5	678.9
N of employees	301	1,088	678.9
Mean daily wage, Euro	100.8	54.6	670.3
Three-year backward average citation productivity ($\bar{z}_{et}^{(3)}$)	0.13	0.66	465.6
Total inventor inflows	0.40	3.0	678.9
Inventor productivity growth	-0.01	1.08	13.6
Knowledge-diffusion inflow term	0.13	0.23	13.6
Net inventor inflow rate	0.04	0.13	13.6

SIAB Provides the Worker-Level Comparison Sample

- **Sample of Integrated Labour Market Biographies (SIAB)**
  - Linked employer-employee data in Germany
  - 2% random sample of workers
  - 1980–2019
  - 3,322,316 individuals
- Used to compare inventor-proxy workers with other job-to-job movers
- The comparison sample includes worker characteristics and origin and destination establishment information

Five Occupation Groups Define the Inventor-Proxy Sample

Rank	KldB group	Inventor-years	Share (%)	Cum. share (%)	Univ. ent. (%)
1	25	669,334	19.32	19.32	66.50
2	27	662,456	19.13	38.45	65.68
3	41	550,441	15.89	54.34	75.73
4	26	550,112	15.88	70.22	79.52
5	71	357,378	10.32	80.54	74.18

- The five largest KldB 2010 parent occupation groups account for 80.5% of INV-BIO inventor-years with occupation observed
- We define inventor-proxy workers in SIAB as workers in groups 25, 26, 27, 41, or 71 who also have a university-entrance qualification
- The education restriction keeps the proxy closer to patent-listed inventors while sacrificing coverage
- Measurement error in the proxy should attenuate differences between inventor-proxy and other workers

Comparing SIAB Inventor Proxies with INV-BIO Inventors

Summary statistics		SIAB		INV-BIO
(1980–2014)		Workers	Inventor-proxy	Inventors
Daily wage, Euro	Mean	63.8	99.5	123.8
	S.D.	45.5	55.6	52.1
Age	Mean	38.5	39.0	38.9
	S.D.	12.6	11.2	10.6
Females, %		44.2	33.7	6.4
University-entrance qualification, %		15.3	100.0	70.3
N of observations, thousand		17,474	1,030	3,517

Notes: Inventor-proxy workers are workers whose observed KldB 2010 parent occupation group is 25, 26, 27, 41, or 71 and who have a university-entrance qualification.

Back: SIAB result

SIAB Job-to-Job Sample: Summary Statistics

Worker-level variables	Mean	S.D.	N, thousand
D_{it}^{wage}	0.45	–	652
$\Delta \log w_{it}$	0.173	0.657	652
Wage increase indicator	0.74	–	652
$I_{i,t-1}$	0.10	–	652
Age	33.3	11.3	652
Female, %	42.7	–	652
University-entrance qualification, %	27.0	–	652
Log daily wage	3.877	0.871	652

Inventor-Proxy Results Are Robust Among Wage-Increase Moves

	Full job-to-job sample		Wage-increase moves	
Dependent variable	Lower-mean-wage dest.	Own-wage growth	Lower-mean-wage dest.	Own-wage growth
	(1)	(2)	(3)	(4)
Inventor-proxy mover	0.0080***	0.0581***	**0.0094******	**0.0535******
	(0.0021)	(0.0033)	(0.0025)	(0.0027)
Demographic controls	Yes	Yes	Yes	Yes
Prior wage controls	Yes	Yes	Yes	Yes
Origin and destination FE	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes
Observations	651,554	651,554	404,722	404,722

- The lower-mean-wage-destination coefficient is slightly larger in the restricted sample
- The own-wage-growth coefficient remains close to its full-sample estimate

Back: SIAB result

Inventor Flow Robustness: Wage-Increase Moves

Rank measure	Sample	Not higher	Higher
Citation/inventor	All moves	53.6	46.8
Citation/inventor	Wage-increase moves	52.2	47.4
Mean wage	All moves	52.0	48.0
Mean wage	Wage-increase moves	50.7	49.4

- Entries are row averages from the decile transition matrices
- Not higher means the destination rank bin is not above the origin rank bin
- The pattern survives after restricting to moves in which the inventor's own wage increases

Back: flows

German Inventor Mobility Rates: 1999–2007

Year	Inventor-years	Job-to-job moves	Rate (%)
1999	117,904	14,358	12.2
2000	126,076	15,790	12.5
2001	129,050	16,194	12.6
2002	130,510	13,301	10.2
2003	130,597	11,180	8.6
2004	129,933	11,611	8.9
2005	129,425	10,093	7.8
2006	129,400	10,133	7.8
2007	129,446	10,699	8.3

- Mobility fell sharply after 2001, before the late-2000s slowdown
- The decline counterfactual uses the 2001 and 2007 rates as early and middle benchmarks

◂ Back: mobility decline

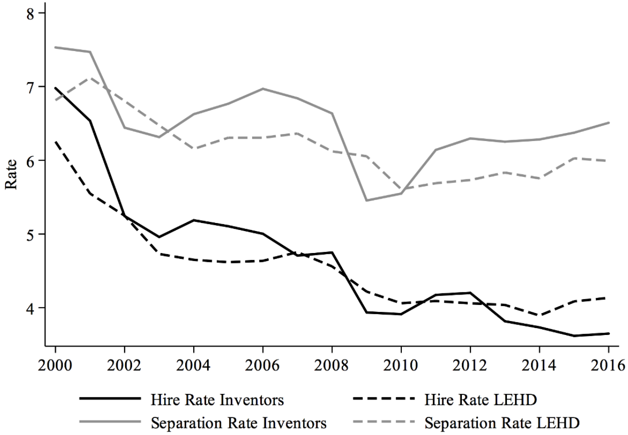
German Inventor Mobility Rates: 2008–2014

Year	Inventor-years	Job-to-job moves	Rate (%)
2008	129,127	10,340	8.0
2009	127,244	7,773	6.1
2010	124,402	7,146	5.7
2011	122,456	7,260	5.9
2012	120,339	8,130	6.8
2013	118,021	6,998	5.9
2014	115,710	5,531	4.8
Total	2,009,640	166,537	8.3

- The final decline counterfactual targets the 2014 rate, 4.8%
- Moves require an establishment change with no intervening unemployment spell

Back: mobility decline

U.S. Inventor Mobility: External Benchmark



Source: Akcigit and Goldschlag (2023), based on Inventor Employment History; comparison sample: 1% LEHD.
External benchmark; not used in the German model calibration.

Back: mobility decline

The Regression Separates Diffusion from Internal R&D

We use establishment productivity growth to identify diffusion from inflows and internal R&D from incumbent productivity. Let $z = x^{-1} = k/n$ denote establishment inventor productivity. Under

$$\alpha(x, x') = \overline{\alpha}/x', \qquad \mu(x) = \overline{\mu}x^{1/2},$$

the estimating equation is

$$\Delta \log z^{\text{inc}}_{e,t} = \overline{\alpha} \underbrace{\sum_{e'} \frac{m_{e' \to e,t}}{n_{e,t-1}} \frac{z_{e',t-1}}{z_{e,t-1}}}_{\text{diffusion exposure}} + \overline{\mu} \underbrace{\Delta z^{-1/2}_{e,t-1}}_{\text{internal R\&D exposure}} + \underbrace{\text{Controls}_{e,t}}_{\text{controls}} + \varepsilon_{e,t}.$$

- $\overline{\alpha}$ pins down diffusion from source-establishment productivity carried by inflows
- $\overline{\mu}$ pins down internal R&D from lagged incumbent productivity
- Incumbent-inventor outcomes avoid mechanically counting entrants' own patent output

Back: calibration

Functional Forms for Search, R&D, and Diffusion

Function	Description
$c(v) = \frac{\bar{c}}{\phi+1} v^{\phi+1}$	Vacancy cost function
$q(\theta) = \bar{q}\theta^{\eta-1}$	Matching function
$\mu(x) = \bar{\mu} x^{\epsilon}$	Internal R&D function
$\alpha(x, x') = \overline{\alpha}/x'$	Knowledge diffusion function

Parameters: Fixed and External Values

Parameter	Value	Description	Source or target
ρ	0.004	Discount rate	5% annual real interest rate
$\bar{c}$	1	Scalar of vacancy cost	Normalization
ϕ	3.45	Curvature of vacancy cost	Bilal et al. (2022)
η	0.50	Elasticity of matches	Petrongolo and Pissarides (2001)
ϵ	0.50	Innovation elasticity	Acemoglu et al. (2018)
π	0.23	Instantaneous profit	Markup is 1.3; Cavenaile et al. (2023)
N	1.95	Mass of inventors	Average German inventor productivity
β	0.50	Bargaining power	Symmetric Nash benchmark

Parameters: Estimated Values and Targets

Parameter	Value	Description	Source or target
$\overline{\alpha}$	0.526	Knowledge diffusion scale	Estimated from productivity growth
$\overline{\mu}$	0.0212	Internal R&D scale	Estimated from productivity growth
σ	0.203	R&D shock s.d.	Estimated from productivity growth
λ	1.053	Innovation step size	2% annual growth rate
$\overline{q}$	0.0198	Matching efficiency	12.6% inventor job-flow rate

Annual-to-Monthly Calibration Conversions

The regression is annual, while one model period is one month.

$$\overline{\alpha} = 0.526, \qquad \overline{\mu} = \frac{0.2546}{12} = 0.0212, \qquad \sigma = \frac{0.7020}{\sqrt{12}} = 0.203.$$

$$\iota = \frac{\log(1 + 0.126)}{12}, \qquad g = \frac{\log(1 + 0.02)}{12}, \qquad \lambda = \exp\left(\frac{g}{\delta}\right) = 1.053.$$

- ι is the monthly log equivalent of the 2001 annual inventor job-to-job transition rate
- g is the monthly log equivalent of a 2% annual aggregate growth rate
- $\overline{q}$ is chosen so the model matches ι at the baseline

Back: calibration

Productivity Growth Identifies Diffusion and Internal R&D

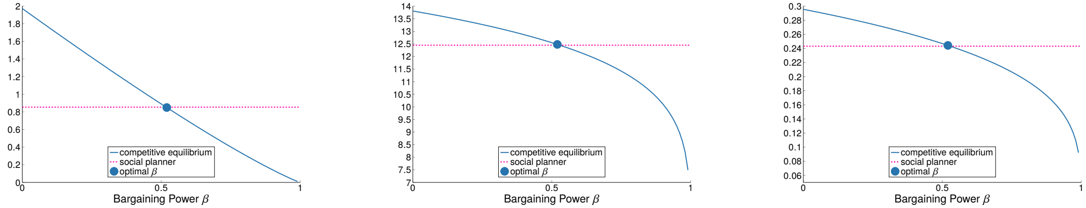
	Inventor-productivity growth		
	(1)	(2)	(3)
Diffusion inflow term	0.526***	0.306***	0.250***
	(0.071)	(0.073)	(0.089)
Inverse-root lagged productivity	0.255***	0.347***	0.440***
	(0.023)	(0.024)	(0.029)
Establishment FE	✓	✓	✓
Year FE	✓		
Industry × region × year FE		✓	✓
Establishment-specific linear trend			✓
Root mean square error	0.702	0.641	0.653
N	12,690	10,812	10,812

The dependent variable is incumbent inventor-productivity growth; newly hired inventors' own patent output is not mechanically included in the outcome. SEs are in parentheses. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

Back: calibration

Weaker Non-Competes Raise Vacancy Posting and Tightness

(a) Vacancy costs **(b)** Job-to-job mobility **(c)** Market tightness

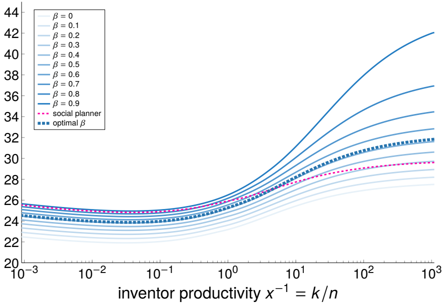


- Lower β raises the poacher's private return, increasing vacancy posting, labor market tightness, and mobility
- The same force that raises diffusion also raises recruiting resource costs

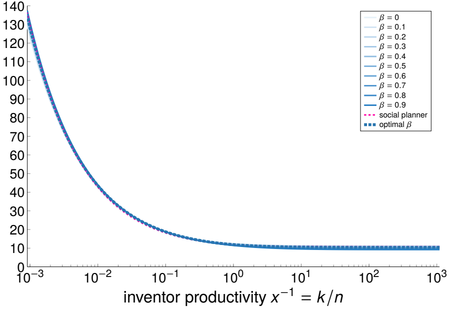
▸ Marginal values

Bargaining Power Mainly Moves the Inventor Value

(a) Inventor value



(b) Knowledge value

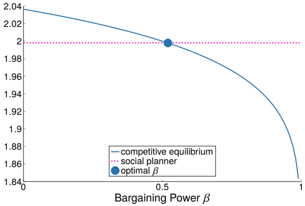


- Left: $\Omega_n(k,n) = S_x(x)$; right: $\Omega_k(k,n) = S(x) - xS_x(x)$
- Bargaining power affects the hiring value more strongly than the knowledge value

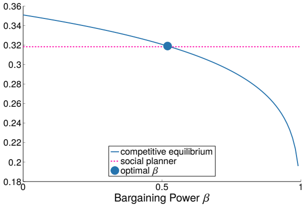
Back: ban

Across Bargaining Power, Growth Responds Mainly Through Diffusion

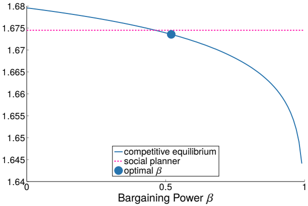
(a) Total growth



(b) Diffusion margin



(c) Internal R&D margin



- Diffusion is only 16.2% of total growth, but it is the more elastic margin

Vacancy Costs Enter the CE Welfare Calculation

The resource constraint maps output and vacancy costs into consumption:

$$\log C(t) = \log Y(t) + \log\left\{1 - \int c(v(x,t))\, dF(x,t)\right\}.$$

Given a baseline consumption path $\{C(t)\}$ and a counterfactual path $\{\widetilde{C}(t)\}$, the consumption-equivalent welfare change is

$$\mathcal{L} = \exp\left[\rho\left\{V(\{\widetilde{C}(t)\}) - V(\{C(t)\})\right\}\right], \qquad \Delta^{CE} = 100(\mathcal{L} - 1).$$

$$V(\{C(t)\}) = \int_0^\infty e^{-\rho t} \log C(t)\, dt.$$

- Non-compete ban: higher growth but higher vacancy costs, so $\Delta^{CE} = -0.38\%$
- Mobility decline: lower growth and weaker diffusion, so $\Delta^{CE} = -3.49\%$ relative to the initial BGP

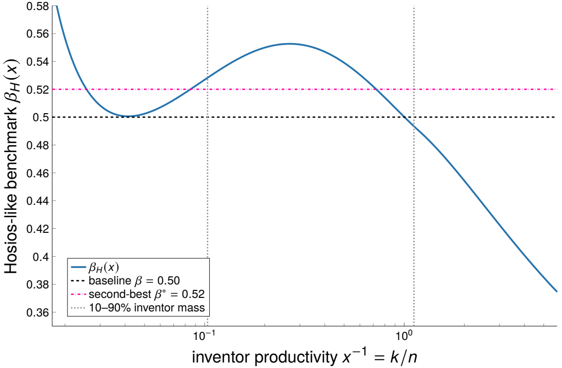
◂ Back: ban   ◂ Back: mobility decline

A Fixed Split Cannot Decentralize the Planner Everywhere

Equating private and planner vacancy-posting margins implies a state-dependent planner-implied bargaining power:

$$\beta_H^P(x,t) = -\frac{\theta q'(\theta)}{q(\theta)} \frac{B^P(t)}{A^P(x,t)},$$

$$= (1-\eta)\frac{B^P(t)}{A^P(x,t)}.$$



◂ Back: wedges ◂ Back: split

Formal Environment

Household and final good

The representative household supplies production labor L and inventor labor N inelastically and has utility

$$\int_0^\infty e^{-\rho t} \log C(t)\, dt.$$

Final output is Cobb–Douglas across a unit continuum of products:

$$\log Y(t) = \int_0^1 \log\{z(\omega,t) l(\omega,t)\}\, d\omega.$$

Products and growth

Product quality follows a quality ladder,

$$z(\omega,t) = \lambda^{m(\omega,t)}, \qquad \lambda > 1.$$

If each product faces creative destruction at rate $\delta(t)$, then

$$\log Y(t) = \log \lambda \int_0^t \delta(\tau) d\tau + \log L,$$

$$\frac{d \log Y(t)}{dt} = \delta(t) \log \lambda.$$

Each product line yields instantaneous profit π.

Back: model

Firm State and Knowledge Dynamics

A firm owns product lines, or knowledge capital, k, and employs inventor mass n. Define

$$x \equiv n/k, \qquad x^{-1} = k/n.$$

Internal R&D and destruction

$$d\log k|_{R\&D} = \left\{\mu(n/k) - \frac{\sigma^2}{2}\right\} dt + \sigma dW(t)$$

$$d\log k|_{\text{destruction}} = -\delta(t)dt$$

$\mu(\cdot)$ is increasing and concave, so R&D output per inventor is increasing in k/n.

Knowledge diffusion

When a (k, n) firm hires inventors from a (k', n') firm,

$$\frac{dk|_{\text{from }(k',n')}}{dn|_{\text{from }(k',n')}} = \hat{\alpha}(k/n, k'/n').$$

In product-line state notation,

$$\alpha(x, x') \equiv \hat{\alpha}(1/x, 1/x').$$

Search Setup: Meetings, Contracting, and Sampling

Meetings

A firm posts $\hat{v}$ vacancies at cost $c(\hat{v}/k)k$, where $v \equiv \hat{v}/k$. Let V denote total vacancies and N inventor mass:

$$\theta(t) = V(t)/N, \qquad q(\theta) = m(N, V)/V.$$

A vacancy meets an inventor at rate $q(\theta)$; an inventor meets a vacancy at rate $\theta q(\theta)$.

Contracting

- Firms choose vacancies $v(x, t)$ to maximize the joint value of the firm and incumbent inventors
- A move occurs only when joint poaching surplus is positive: $\mathbb{1}_P(x, x', t) = \mathbb{1}\{M(x, x', t) > 0\}$
- Conditional on a move, the poacher receives share $1 - \beta$; the source firm and inventor receive share β

Sampling measures.

$$1 = \int f(x,t)dx, \qquad N = \int xf(x,t)dx, \qquad V(t) = N\theta(t) = \int v(x,t)f(x,t)dx,$$
$$f_x(x,t) = \frac{xf(x,t)}{N}, \qquad f_v(x,t) = \frac{v(x,t)f(x,t)}{V(t)}.$$

Firm-Level Joint-Value HJB

Let $\Omega(k,n,t)$ denote the joint value of a firm with knowledge capital k and inventor employment n. Its HJB is

$$\rho\Omega(k,n,t) - \Omega_t(k,n,t) = \max_{\hat{v}\geq 0}\Big\{\pi k - c(\hat{v}/k)k + [\mu(n/k) - \delta(t)]k\Omega_k(k,n,t) + \frac{\sigma^2}{2}k^2\Omega_{kk}(k,n,t)$$
$$+ (1-\beta)q(\theta(t))\hat{v}\widehat{A}^{\mathrm{in}}(k,n,t) + \beta\theta(t)q(\theta(t))n\widehat{A}^{\mathrm{out}}(k,n,t)\Big\}.$$

$$\widehat{A}^{\mathrm{in}}(k,n,t) = \int \widehat{M}((k,n),(k',n'),t)\, dG_n(k',n',t),$$

$$\widehat{A}^{\mathrm{out}}(k,n,t) = \int \widehat{M}((k',n'),(k,n),t)\, dG_{\tilde{v}}(k',n',t).$$

$$\widehat{M}((k,n),(k',n'),t) = \left[\Omega_n(k,n,t) + \hat{\alpha}(k/n,k'/n')\Omega_k(k,n,t) - \Omega_n(k',n',t)\right]^+.$$

Product-Line State Reduction

Let $\Omega(k,n,t)$ be the joint value of a firm and its inventors. Homogeneity implies a one-dimensional product-line value:

$$\Omega(k,n,t) = kS(x,t), \qquad x = n/k.$$

The relevant marginal values are

$$\Omega_n(k,n,t) = S_x(x,t), \qquad \Omega_k(k,n,t) = S(x,t) - xS_x(x,t),$$

$$\Omega_{kk}(k,n,t) = \frac{1}{k}x^2 S_{xx}(x,t).$$

For a destination state x and source state x', product-line match surplus is

$$M(x,x',t) = \left[S_x(x,t) + \alpha(x,x')\{S(x,t) - xS_x(x,t)\} - S_x(x',t)\right]^+.$$

◂ Back: diffusion

Homogeneity Caveat: Fixed-Size Product-Line Jumps

Proportional shocks

The state reduction uses continuous knowledge capital:

$$\Omega(k, n, t) = kS(x, t), \qquad x = n/k.$$

A shock that scales with k keeps terms degree one:

$$\frac{\sigma^2}{2} k^2 \Omega_{kk} = k \frac{\sigma^2}{2} x^2 S_{xx}.$$

Fixed-size jumps

If one product line is added and inventor employment is fixed,

$$\Delta_k^+(x, t) = (k+1) S\left(\frac{x}{1 + 1/k}, t\right) - kS(x, t).$$

The normalized HJB contains Δ_k^+, which depends on both x and k.

Interpretation: Gibrat's law

- **Weak**: only mean growth is size-independent: $E[d \log k \mid x, k] = E[d \log k \mid x]$.
- **Strong**: the full growth-rate distribution is size-independent.
- **Proportional shocks**: satisfy strong Gibrat, so $x = n/k$ closes the HJB.
- **Fixed-size jumps**: satisfy only weak Gibrat; Δ_k^+ depends on k and x.

Competitive Equilibrium: Definition

A competitive equilibrium is

$$\{S(x,t), v(x,t), \mathbb{1}_P(x,x',t), f(x,t), \delta(t), \theta(t)\}_{t\geq 0}$$

such that, given $f(x,0)$ with

$$1 = \int f(x,0)dx, \qquad N = \int xf(x,0)dx,$$

1. $S(x,t)$ satisfies the HJB equation
2. $v(x,t)$ satisfies the vacancy-posting first-order condition
3. $\mathbb{1}_P(x,x',t) = 1$ if and only if $M(x,x',t) > 0$
4. $f(x,t)$ follows the Kolmogorov-forward equation
5. $\delta(t)$ aggregates internal R&D and poaching-based diffusion
6. $\theta(t)$ clears the aggregate vacancy market

Competitive Equilibrium: Stationary HJB

Stationary notation suppresses time. Equilibrium objects are the joint value $S(x)$, vacancy policy $v(x)$, poaching rule $\mathbb{1}_P(x, x')$, distribution $f(x)$, creative-destruction rate δ, and tightness θ.

$$\rho S(x) = \pi - c(v(x)) + \{\mu(x) - \delta\}\{S(x) - xS_x(x)\} + \frac{\sigma^2}{2}x^2 S_{xx}(x)$$
$$+ (1-\beta)q(\theta)v(x)A^{\mathrm{in}}(x) + \beta\theta q(\theta)xA^{\mathrm{out}}(x),$$

$$A^{\mathrm{in}}(x) = \int M(x, x')dF_x(x'), \qquad A^{\mathrm{out}}(x) = \int M(x', x)dF_v(x').$$

$$M(x, x') = \left[S_x(x) + \alpha(x, x')\{S(x) - xS_x(x)\} - S_x(x')\right]^+ .$$

Competitive Equilibrium: Vacancy Posting and Mobility

Vacancy posting equates marginal cost to the expected employer share of match surplus:

$$c'\left(v\left(x\right)\right) = \left(1-\beta\right) q(\theta) \int M(x, x') dF_x(x')$$

Given a match, the inventor moves if the surplus is positive:

$$\mathbb{1}_P(x, x') = \begin{cases} 1 & \text{if} \quad M(x, x') > 0 \\ 0 & \text{otherwise} \end{cases}$$

Competitive Equilibrium: Stationary Distribution

The product-line distribution over $x = n/k$ follows a KF equation. Define

$$a(x) = \mu(x) - \delta + q(\theta)v(x) \int \mathbb{1}_P(x, x')\alpha(x, x')dF_x(x')$$

and

$$\begin{aligned} b(x) = &- \{\mu(x) - \delta\}x + q(\theta)v(x) \int \mathbb{1}_P(x, x')\{1 - \alpha(x, x')x\}dF_x(x') \\ &- \theta q(\theta)x \int \mathbb{1}_P(x', x)dF_v(x'). \end{aligned}$$

Then the stationary density satisfies

$$0 = a(x)f(x) - \frac{\partial}{\partial x}\{b(x)f(x)\} + \frac{\partial^2}{\partial x^2}\left\{\frac{\sigma^2}{2}x^2 f(x)\right\}.$$

Competitive Equilibrium: Aggregates

Labor market tightness clears the aggregate vacancy market:

$$N\theta = \int v\,(x)\, f\,(x)\, dx$$

The creative-destruction rate aggregates internal innovation and poaching-based knowledge diffusion:

$$\begin{aligned}\delta = &\int \mu(x) dF(x) \\ &+ N\theta q(\theta) \int\int \mathbb{1}_P(x, x')\alpha(x, x') dF_x(x') dF_v(x)\end{aligned}$$

Mobility and resource use are

$$\iota = \theta q(\theta) \int\int \mathbb{1}_P(x, x') dF_x(x') dF_v(x),$$

$$C = Y\left\{1 - \int c(v(x)) dF(x)\right\}.$$

Firm Growth: Knowledge and R&D

$$
\begin{aligned}
d\log x^{-1}(x,t) &= d\log k(x,t) - d\log n(x,t) \\
d\log k(x,t) &= \underbrace{q(\theta(t))v(x,t)\int \mathbb{1}_P(x,x',t)\alpha(x,x')dF_x(x',t)dt}_{\text{knowledge gain from poaching}} \\
&\quad + \underbrace{\left(\mu(x) - \delta(t) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)}_{\text{R\&D minus creative destruction}}.
\end{aligned}
$$

Firm Growth: Inventor Flows

$$d \log x^{-1}(x,t) = d \log k(x,t) - d \log n(x,t).$$

$$d \log n(x,t) = \underbrace{\frac{q(\theta(t))v(x,t)}{x} \int \mathbb{1}_P(x,x',t)dF_x(x',t)dt}_{\text{inventor inflow rate}}$$

$$\underbrace{- \theta(t)q(\theta(t)) \int \mathbb{1}_P(x',x,t)dF_v(x',t)dt}_{\text{inventor outflow rate}}.$$

- Firm growth rises when knowledge gains from poaching exceed inventor inflow dilution and outflows

Innovation Wedge: Planner Value of Quality Steps

The innovation wedge compares the private value of a new product line, based on the profit flow π, with the planner's social value of a quality step.

$$\max \int_0^\infty e^{-\rho t} \left[\log Y(t) + \log \Psi^P(t)\right] dt,$$

$$\Psi^P(t) = 1 - \int c(v^P(x,t)) f^P(x,t) dx.$$

$$\frac{\log \lambda}{\rho} \Psi^P(t) = \int \left[S^P(x,t) - x S_x^P(x,t)\right] f^P(x,t) dx$$

Competitive value

- Product-line payoff is disciplined by π
- Appropriability pushes private value down
- Business stealing pushes private value up

Planner value

- Left side: consumption value of a quality step
- Right side: expected marginal social value of knowledge
- Pins down $\varpi^P(t)$ in the planner HJB

Back: wedges

Search/Matching Wedge: Vacancy Margin

The search/matching wedge compares the poacher's private return with the planner's marginal social value of a vacancy.

Competitive: private posting

$$c'(v^C(x,t)) = (1-\beta)q(\theta^C(t))A^C(x,t),$$
$$A^C(x,t) = \int M^C(x,x',t)dF_x^C(x',t)$$

- Takes $\theta^C(t)$ as given
- Receives only the poacher share $1-\beta$

Planner: social vacancy value

$$c'(v^P(x,t)) = q(\theta^P(t))A^P(x,t) + \theta^P(t)q'(\theta^P(t))B^P(t),$$
$$A^P(x,t) = \int M^P(x,x',t)dF_x^P(x',t),$$
$$B^P(t) = \iint M^P(x'',x',t)dF_x^P(x',t)dF_v^P(x'',t).$$

- Hold-up pushes private vacancy posting down; congestion pushes it up
- Since $q'(\theta) < 0$, extra vacancies reduce other vacancies' meeting rates

Back: wedges Back: policy